Assignment 3, due September 30

Corrections: (None yet. See message board)

1. (*a hard exercise in calculus*) Consider the equation $e^{\lambda x} = 1 + x$. Suppose $0 < \lambda \leq 1$. Show that for $\lambda = 1$, the only solution is $x = 0$. Show (by picture if necessary) that if $\lambda < 1$ there is another solution $x(\lambda) > 0$. Suppose $\lambda < 1$ but λ is close to 1. Show that there is an approximate formula of the form
$$x(\lambda) \approx C(1 - \lambda) \ .$$
Find C. Hint: Use the first few terms of the Taylor expansion of $e^{\lambda x}$ to get an approximation of $e^{\lambda x}$ that is valid when λx is small. Experiment with the order of the approximation (one term, two terms, three terms, ...) until you get a sensible answer.

2. Suppose U is a random variable that is uniformly distributed in $[0, 1]$. Suppose $\lambda > 0$ and $T = -\frac{1}{\lambda}\log(U)$. Show that the PDF of T is $f(t) = \lambda e^{-\lambda t}$ if $t > 0$, and $f(t) = 0$ if $t < 0$. Make sure to verify both parts of the formula for f. Hint: The definition of probability density is
$$\mathrm{P}(T \in [t, t + dt]) = f(t)dt \ .$$
If $t > 0$, find a formula for u and du so that
$$T \in [t, t + dt] \iff U \in [u, u + du] \ .$$
You find the relationship between du and dt by differentiating the relationship between u and t.

3. The *covariance* between random variables X and Y is
$$\mathrm{cov}(X, Y) = \sigma_{XY} = \mathrm{E}[(X - \mu_X)(Y - \mu_Y)] \ ,$$
where μ_X is the expected value of X and μ_Y is the expected value of Y. The *correlation coefficient* is
$$\mathrm{corr}(X, Y) = \rho_{XY} = \frac{\sigma_{XY}}{\sqrt{\sigma_X^2 \sigma_Y^2}} \ .$$
Here, $\sigma_X^2 = \mathrm{E}\left[(X - \mu_X)^2\right]$ is the *variance* of X. It is known that $|\rho_{XY}| \leq 1$ for any pair of random variables. Moreover, $|\rho_{XY}| = 1$ only if $X = aY + b$ (X is a linear function of Y, equivalently, Y is a linear function of X). If Y is non-linear function of X, then $|\rho_{XY}| < 1$. Investors often invest in a combination of assets with returns that have low correlation.

(a) Suppose S is a risky asset whose value is uniformly distributed in the interval $[1,3]$. Write a formula for $f(x)$, which is the PDF of S and calculate its mean (obvious, once you have it) and variance.

(b) Suppose C is a call option on S with strike price $K = 2$. Write a formula for $C(S)$. Use this formula to calculate the mean and variance of C.

(c) Calculate ρ_{SC}. Show that this number is larger than zero (pretty obvious) and less than one. That might seem surprising. It is possible for the value of C to be completely determined by the value of S, and yet the correlation coefficient is not equal to one. Chapter 3 of the textbook explains why this can make people want to put options in a portfolio.

4. Let Y be the total payment for a risky bond with a principal P and a coupon rate R. This bond defaults at a random time T. The maturity time of the bond is t_M. If $T > t_M$, then the bond makes continuous coupon payments in the amount Rdt in every interval of time $[t, t+dt]$ for $t < t_M$, and the bond makes a principal payment P at time t_M. If the bond defaults at time $T < t_M$, then only the payments before the default time T are made. We model the default time as an exponential random variable with rate constant λ. We model different bond defaults as independent[1]. Suppose we have a basket of L independent risky bonds with the same maturity date, principal, and coupon rate. The total payment for the basket is

$$X = Y_1 + \cdots + Y_L \ .$$

The individual payments are either $Y_j = Rt_M + P$, if $T_j > t_M$, or $Y_j = RT$, if $T_j < t_M$. This formula omits present value calculations. It is wrong to leave out discounting, but it makes the problem simpler.

Write a script in `R` that makes a histogram that represents the PDF of the random variable X. Simulate n independent X values. Choose n so that it takes your computer a few seconds to run the script. Experiment with a few different values of the parameters L, λ, t_M, etc., to get a few histograms that look different. When L is large and the default probability is high,

$$\Pr(\text{default}) = \Pr(T < t_M) = 1 - e^{-\lambda t_M} \ ,$$

the histogram should look roughly Gaussian. If the default probability is low or if L is not large, the histogram is less "normal". There may be a big spike in the histogram corresponding to no defaults.

You can write the script so that the parameters are set in the script rather than being entered by the user when the script runs. That means statements in the script like `L = 10`. This will make it less tedious for you

[1] This assumption is not only wrong, it was a big part of the misunderstanding of default risk that led to the financial crisis of 2007-2008. We will discuss more realistic models of bond default later in the class.

to run the script many times. Hand in a printout of the script and a few interesting histograms. The histograms should be self-explanatory, parameter values listed in the histogram title. There is sample code and explanations of it for you to use in creating your script.