Assignment 4, due October 7

Corrections: (None yet. See message board)

1. Suppose $f(x,y) = 2x^2 + 2xy + y^2$ and $g(x,y) = x^2 + y^2$. This exercise calculates the maximum and minimum of f on the circle defined by $g = 1$. Terminology for the problem:

$$f_{\text{min}} = \min_{g(x,y)=1} f(x,y)$$
$$f_{\text{max}} = \max_{g(x,y)=1} f(x,y)$$
$$(x_{\text{min}}, y_{\text{min}}) \text{ minimizes} f(x,y) \text{ with } g(x,y) = 1, \text{ also written}$$
$$(x_{\text{min}}, y_{\text{min}}) = \arg\min_{g=1} f \quad (\arg\min \text{ is the } \textit{argument} \text{ that } \textit{minimizes} \text{ ...})$$
$$(x_{\text{max}}, y_{\text{max}}) = \arg\max_{g=1} f$$

   (a) Calculate ∇f and ∇g.

   (b) Make a large graph (at least half a page) with a big unit circle and ∇f indicated (as an arrow with the right length and direction) at several points on the unit circle, including the horizontal and vertical points (intersection with the horizontal and vertical axes) and some others. Use this diagram to explain, for example, how you can move (x,y) away from $(1,0)$ while remaining on the unit circle and make f a little larger than $f(1,0)$ or a little smaller.

   (c) Write the system of two equations that corresponds to the vector equation $\nabla f = \lambda \nabla g$. The unknowns in this system are x, y, and λ.

   (d) If λ is known, the equations are linear in x and y. Use Gaussian elimination to eliminate x and get one equation involving λ and y.

   (e) We know from part (b) that $y \neq 0$ at the constrained maximum or minimum. Use this fact to find the two possible values of λ.

   (f) Take one of the equations from part (c), together with $x = \pm\sqrt{1-y^2}$ to derive a formula for y in terms of λ. Derive a similar formula for x in terms of λ. Note that for each value of λ there are two (x,y) pairs. Indicate these on your diagram from part (b). Write the two possible $(x_{\text{min}}, y_{\text{min}})$ and the two possible $(x_{\text{max}}, y_{\text{max}})$. Find the values f_{min} and f_{max}.

   (g) Show that for any of the choices, $(x_{\text{min}}, y_{\text{min}})$ is perpendicular to $(x_{\text{max}}, y_{\text{max}})$.

(h) Show that the two equations $\nabla f = \lambda \nabla g$ do not determine x or y.

2. Consider instead the problem of maximizing g subject to the constraint $f = 1$ (f and g as in problem (1)).

   (a) Show that the set of (x, y) with $f(x, y) = 1$ is an ellipse.

   (b) Show that maximizing g corresponds to the geometry problem of finding the point on this ellipse farthest from the origin. (This distance is called the *semi-major axis* because it is half of the *major axis*, which is the greatest width of the ellipse.)

   (c) Use the Lagrange multiplier method of problem (1) to find values (x, y, λ) with $\nabla g = \lambda \nabla f$ and $f(x, y) = 1$. Show that the (x, y) pairs you get this way are multiples of the $(x_{\min}, y_{\min})$ or $(x_{\max}, y_{\max})$ pairs from problem (1).

   (d) Show that minimizing f with a constraint on g yields points that also can be found by maximizing g with a constraint on f. (In mean variance, you can minimize risk with a constraint on expected return, or you can maximize expected return with a constraint on risk.)

3. Consider the matrix and vector

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad , \qquad u = \begin{pmatrix} x \\ y \end{pmatrix} \ .$$

   (a) Show that $u^t A u = ax^2 + 2bxy + cy^2$, and that $u^t u = x^2 + y^2$. Here $u^t = (x, y)$ is the row vector that is the transpose of the column vector u.

   (b) Show that

$$\nabla \left[u^t A u \right] = A u \ .$$

   (c) Show that this is not true in general if A is not a symmetric matrix.

   (d) Use the result of part (b) to show that if A is a symmetric matrix, then maximizing $u^t A u$ with the constraint $u^t u = 1$ leads to the eigenvalue equation

$$Au = \lambda u \ .$$

   (e) Show that the maximum of $u^t A u$ with $u^t u = 1$ is equal to $\lambda_{\max}$.

   (f) Show that if

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \ ,$$

   then the equations of (1c) are the same as the equations of part (d) here. Show that the equation of (1e) is the same as the eigenvalue equation

$$\det\left(A - \lambda I\right) = \det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 1 - \lambda \end{pmatrix} = 0 \ .$$

4. Let X be normal with mean zero and variance σ^2. Let Y be a call option on X struck at $K = 0$. Calculate the 2×2 covariance matrix of (X, Y).

5. Suppose X is normal with mean μ and variance σ^2, which is small. Let $Y = e^X$. It seems intuitive that $\mathrm{E}[Y] \approx e^\mu$, because X is likely to be close to μ. Find an approximate expression for $\mathrm{E}[Y] - e^\mu$ that involves a power of σ. Hint: Write $X = \mu + \varepsilon$, where ε is small (with high probability) and a Gaussian random variable. Expand e^X in a Taylor series of the appropriate order (the lowest order that gives a non-zero correction to e^μ) in ε. You will find that $e^\varepsilon \approx 1 + \varepsilon$ is not enough.

6. Write a script in `R` that solves the mean/variance portfolio optimization problem

$$\sigma^2(d) = \min w^t C w \ \ , \quad \sum_{k=1}^{n} w_k = 1 \ \ , \quad \mu^t w = d \ .$$

   Here, C is an $n \times n$ symmetric positive definite matrix and μ is a column vector with n entries. The data is

   - The matrix C
   - The number of risky assets, n
   - The vector of expected returns, μ
   - The desired expected return, d

   These can be set in the script using assignment statements, as in the file `LinearAlgebra.R`. Run your script for various values of d and use them to make a rough plot, by hand on paper, of the efficient frontier. You can make up the data, but it must have $n \geq 3$ and C must be symmetric and positive definite. One example is the matrix A is `LinearAlgebra.R`, but feel free to use a bigger one or a different one.