Assignment 6, due November 11

Corrections: (Typos in problem 1 corrected, the $<$ corrected to $>$ in problem 2. See message board)

1. Consider the two state one period model with actual probabilities:

$$S_{t+\Delta t} = \begin{cases} uS_t & \text{with probability } p_u \\ dS_t & \text{with probability } p_d \;. \end{cases} \tag{1}$$

These are not the risk neutral probabilities, which would be written q_u and q_d. The binomial tree model uses parameters μ, the expected return rate, and σ, the *volatility*. The parameters in the two state model are written as $u = 1 + \varepsilon$, $d = 1 - \varepsilon$, $p_u = \frac{1}{2} + \delta$, $p_d = \frac{1}{2} - \delta$. Note that we have assumed a relation between u and d. This is done for convenience. But now, the parameters in the one period model are ε and δ. These are chosen so that the following relations hold:

$$\mathrm{E}_p[\Delta S] = \mu S_t \Delta t \tag{2}$$

$$\mathrm{var}_p(\Delta S) = \sigma^2 S_t^2 \Delta t \tag{3}$$

Here, $\Delta S = S_{t+\Delta t} - S_t$ is the change in the asset price over the period. The p subscript in E_p is a reminder that we are using the "p measure" (the real world probabilities) here.

Find formula approximate formulas for ε and δ in terms of the parameters μ, σ, and Δt. To get the approximation, express (2) and (3) in terms of ε and δ. Among other equations, you should get something like $\varepsilon^2 + 4\varepsilon^2\delta^2 = \sigma^2 \Delta t$. If δ is small, the second term on the left is much smaller than the right term. If you leave out the smaller term, you get a simple formula for ε in terms of Δt and σ. Use this simple formula rather than the correct but more complicated one to solve for δ. Show that your simple formulas give ε and δ that satisfy (2) and (3) up to errors of order Δt^2.

2. Consider a three period model with two binary "shocks". In this model, the formulas (1) apply at time $t + \Delta t$ also. For example,

$$\Pr(S_{t+2\Delta t} = udS_t) = 2p_u p_d \;.$$

Consider the change in the asset price after two shocks, which is $S_{t+2\Delta t} - S_t$. Show that

$$\mathrm{E}_p[S_{t+2\Delta t} - S_t] > 2\mu S_t \Delta t \;.$$

Calculate the difference, to *leading order* in Δt. That means: figure out the power of Δt that makes the largest contribution to the difference, and find the coefficient of that power of Δt. For example, if the difference were $\sqrt{9\Delta t + 8\Delta t^2}$, the answer to this question would be $3\sqrt{\Delta t}$.

3. Take the formulas for for u and d in terms of σ and Δt that were found in problem (1). Find formulas for approximate q_u and q_d in terms of σ, Δt, and r (the risk free rate). Your approximate q_u and q_d should agree with the exact ones up to order $\Delta t^{3/2}$. (Hint: Use the appropriate Taylor approximation of $e^{r\Delta t}$.) Show that with these u, d, and q_u and q_d, the formulas (2) and (3) are still satisfied, but with μ replaced by r.

   (This is an important principle in option pricing theory. In the risk free measure, the expected return on a risky asset is the same as the expected return of the risk free asset. A risk neutral investor only cares about expected return, not risk. A risk averse investor would not buy S_t (the risky asset) unless $\mu > r$, the return on the risky asset is larger than the risk free return.)

4. Consider the three asset allocation problem from the midterm, with (note the change in w)

$$\mu = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad , \qquad C = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \quad , \qquad w = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} .$$

   Show that this allocation is inefficient. Find a Δw that satisfies the budget constraint that, according to first order Taylor calculations (linear in Δw) reduces σ_P^2 without changing μ_P. Write a script in `R` to compare your predicted decrease in σ_P^2 to the actual decrease. You may save time by downloading the code `PortPlay.R`. As it is, the result should be

```
> source("PortPlay.R")
portfolio variance is 2
portfolio expected return is 1.5
predicted change in variance is  0.01 , actual change is  0.0204
```

   You will see that the predicted and actual change in σ_P^2 are pretty different. That's because there is a mistake in the code.

5. This exercise uses stock price data to estimate a covariance matrix and expected return. Download and run the script `StockDownload.R`. The result should be:

```
> source("StockDownload.R")
3  tickers
ticker  T  has L =  7892
   most recent prices are  33.48 33.63 33.61 33.51 33.55
ticker  BA  has L =  13555
   most recent prices are  148.19 148.09 148.4 148.07 147.18
ticker  CSCO  has L =  6456
   most recent prices are  28.47 28.61 28.77 28.85 29.14
Ticker  T  has mean daily return  0.0001505585
Ticker  BA  has mean daily return  0.0003551018
Ticker  CSCO  has mean daily return  0.0006215961
C_11 = var( T ) is  0.0003685396
C_12 = cov( T , BA ) is  8.841097e-05
The correlation coefficient is  0.2035745
```

The each of the tasks below will lead you to add code to this script. Always do the tasks by adding code to the script. Never do a task by typing at the `R` command line. This way, you can build up a script that does quite a lot automatically.

(a) Add to the script some code that computes the $n \times n$ covariance matrix. Use a single loop (`for (i in 1:n){` $\cdots$) for the diagonal entries, which are the variances. Use a double loop for the $i \neq j$ entries C_{ij}. The loop should run over i values from 1 to $n-1$ and over j values larger than i and not larger than n. Once you have C_{ij} for $j > i$, which is an entry above the diagonal, you can use $C_{ji} = C_{ij}$ to get the corresponding entry below the diagonal. This avoids computing the same covariance twice.

(b) Add code to find efficient portfolios using mean/variance analysis. Find the minimum variance portfolio and the efficient portfolio that has the same expected return as the equal weighted portfolio. See how much the variance of the efficient portfolio differs from the variance of the equal weighted portfolio.

(c) Construct a ticker set of your choice (at least 4). These could be indices such as the S&P500, major international indices (China, Europe, Asia, developing, developed, ...), or indices of other asset classes such as bonds, real estate, etc. Do some research to find tickers that represent these indices that have reasonably long histories. They could be stocks of your choosing that represent sectors or kinds of companies you are interested in. They could represent combinations. State your interest or interests and the say how the tickers you chose represent that interest or interests.

(d) Your code may use more returns to compute C_{12} than to compute, say, C_{34}. Statisticians might tell you it's better to use the same length number of returns for each covariance. Modify your code to do this. There are two ways to do this. You could find the length of the shortest sequence and use that for all, or you could specify a length and check that all sequences are at least that long. You should compute the expected returns also over that length. Does this change the results?

(e) Make some comments about the results.