Numerical Methods II, Courant Institute, Spring 2012
http://www.math.nyu.edu/faculty/goodman/teaching/NumMethII/index.html
Always check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Numerical Methods II) before doing any work on the assignment.

Assignment 2, due February 16

Corrections: The second part of (4) was corrected to have y_{n+1} instead of y_n on the right side. Question (3a) was modified to add a formal definition of stability.

1. This question discusses the unstable method that shows why stability theory is important. The differential equation is

$$\dot{x} = f(x) . \tag{1}$$

The one lag linear multistep method is

$$x_{n+1} = a_0 x_n + a_1 x_{n-1} + \Delta t[b_0 f(x_n) + b_1 f(x_{n-1})] . \tag{2}$$

Assume that the time step Δt is fixed. Write $t_n = n\Delta t$ as usual.

   (a) Find equations for the coefficients a_0, a_1, b_0, b_1, that give rise to the method with the highest formal order of accuracy. Since there are four coefficients, we need four equations. These come from plugging the exact solution of (1) into (2) and calculating the leading order of the truncation error. You expand the left and right in Taylor series about t_n. It is convenient to write $x(t_n)$ just as x, $\frac{d}{dt}x(t_n) = x^{(1)}$, etc. For example, $x(t_{n+1}) = x + \Delta t x^{(1)} + \frac{\Delta t^2}{2} x^{(2)} + \frac{\Delta t^3}{6} x^{(3)} + O(\Delta t^4)$. Also, $f(x(t_{n-1})) = f - \Delta t f^{(1)} = f - \frac{d}{dt}\dot{x} = f - \Delta t x^{(2)}$, etc. You get the four equations by setting the coefficients of Δt^k from the left and right sides of (2) equal, $k = 0, 1, 2, 3$.

   (b) Find the roots of the stability polynomial $z^2 = a_0 z + a_1$ and show that at least one of them has $|z| > 1$. Conclude that this method will not work.

2. Derive the third order BDF method that uses x_n, x_{n-1}, and x_{n-2} as well as $f(x_{n+1})$ to predict x_{n+1}. *Third order* means that the local truncation error is of order Δt^4. Is this method zero-stable?

3. Suppose the ODE $\dot{x} = f(x)$ has solution *operator* $x(t) = F(t, x(0))$. Suppose that $\dot{x} = g(x)$ has solution operator $x(t) = G(t, x(0))$. Consider the combined ODE

$$\dot{x} = f(x) + g(x) . \tag{3}$$

A *splitting scheme* constructs an approximate solution of (3) from the exact solution operators F and G or approximations to them.

(a) Consider the scheme

$$\left.\begin{array}{rcl} y_{n+1} & = & F(\Delta t, x_n) \\ x_{n+1} & = & G(\Delta t, y_{n+1}) \end{array}\right\} . \tag{4}$$

Show that this produces a first order accurate (and not second order) and stable scheme for (3). For this problem, and for convergence analysis of time stepping methods for ODE's in general, we use the following definition of stability. Suppose the discrete scheme has the form $x_{n+1} = H(x_n, \ldots, x_{n-p}, \Delta t)$. Suppose $y_{n+1} = H(y_n, \ldots, y_{n-p}, \Delta t) + \Delta t\, r_n$. Then if $x_j = y_j$ for $j = 0, \cdots, p$, then $|x_n - y_n| \leq C(t_n) \sum_{j \leq n} |r_j|$.

(b) Consider the case $x \in \mathbb{R}^2$, $f = \begin{pmatrix} a \\ 0 \end{pmatrix}$ and $g = \begin{pmatrix} 0 \\ b \end{pmatrix}$, where a and b are constants. Draw a picture to show how the splitting method (4) produces motion in the direction $\begin{pmatrix} a \\ b \end{pmatrix}$ out of horizontal motion F and vertical motion G.

(c) Consider the *Strang splitting* scheme

$$\left.\begin{array}{rcl} y_{n+\frac{1}{2}} & = & F(\Delta t/2, x_n) \\ z_{n+\frac{1}{2}} & = & G(\Delta t, y_{n+\frac{1}{2}}) \\ x_{n+1} & = & F(\Delta t/2, z_{n+\frac{1}{2}}) \end{array}\right\} . \tag{5}$$

Show that this produces a second order accurate approximation to (3), but not third order. People who do numerical computing get good at Taylor series manipulations. (If there is a third order splitting scheme, it is complicated and I don't know what it is.)

(d) It seems at first that Strang splitting is more expensive than the first order splitting scheme (4) because there are two applications of F per time step. Show that this is not true. Hint: try to show that $F(\Delta t/2, F(\Delta t/2, x)) = F(\Delta t, x)$, which is because F is a solution to an ODE.

(e) Explain in *psuedocode* (i.e. informal programming language) a scheme that yields overall second order approximations to $x(t_n)$ using the simpler scheme (4), and a well chosen starting procedure, and a special something at time n.

(f) Show that you also get a second order method if you use second order approximations to F and G. These approximations should be second order in the sense that they have third order local truncation error.

4. This exercise takes you through a more modern approach to linear recurrence relations that will be helpful later. Consider a linear recurrence relation with p lage

$$x_{n+1} = a_0 x_n + \cdots + a_p x_{n-p} . \tag{6}$$

If y_n satisfies (6) and $y_j = x_j$ for $j = 0, \ldots, p$, then $y_n = x_n$ for all n (proof by induction on n, first for $n = p+1$, then $n = p+2$, etc.).

(a) There are special solutions of the form $x_n = z^n$. Show that there is a solution of this kind for every value of z that satisfies the equation

$$z^{p+1} = a_0 z^p + \cdots + a_p \; . \tag{7}$$

(b) As in class, define "supervector"

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \\ x_0 \end{pmatrix} ,$$

And show that

$$X_{n+1} = AX_n \quad \text{with} \quad A = \begin{pmatrix} a_0 & a_1 & & \cdots & a_p \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & 0 & & \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} .$$

This matrix A is the *companion matrix* for the recurrence relation (6). Of course, we have $X_n = A^n X_0$.

(c) Show that for each solution of (7) there is an eigenvector of A with eigenvalue z of the form

$$v_z = \begin{pmatrix} z^p \\ z^{p-1} \\ \vdots \\ z \\ 1 \end{pmatrix} .$$

(d) Show that if there are $p+1$ distinct solutions of (7), then the corresponding eigenvectors v_{z_k} for $k = 0, \ldots, p$ form a basis of $\mathbb{C}^{p+1}$. You may use the fact that if V is the matrix whose columns are the vectors v_{z_k} then $\det(V) = \pm \prod_{j \neq k} (z_j - z_k)$.

(e) Still assuming that there are $p+1$ distinct roots, describe or find a formula for the entries of V^{-1} using the *Lagrange interpolation formula*.

(f) Assuming that the roots are distinct and that they satisfy the *root condition* $|z_k| \leq 1$ for all k, show that $\|A^n\| \leq C$ independent of n (i.e. there is a C independent of n so that ...).

(g) Show that if the root condition is satisfied, then the solution of (6) satisfies $|x_n| \leq C\,(|x_0| + \cdots + |x_p|)$, again with C independent of n.

5. **Programming.** None this week so that people can dig out from last week's assignment.