Assignment 4, due March 8

Corrections: (none yet)

1. Let $f(x) = 1/(1 - a\cos(2\pi x))$. This is a smooth periodic function if $|a| < 1$. When a is close to 1 f has a high narrow peak near $x = 0$. Let

$$f^n(x) \;=\; \sum_{\alpha \leq n} \widehat{f}^n_\alpha e^{2\pi i \alpha x}$$

be the trigonometric polynomial interpolating f and $2n+1$ uniformly spaced points $x_j = j/(2n+1)$, $j = 0, \ldots, 2n$. Show that for any a there is a C and a C' so that

$$|f(x) - f^n(x)| \;\leq\; Ce^{-C'n}$$

for all x. Estimate the value of n needed, for a given $a < 1$, to make $|f(x) - f^n(x)| \leq 10^{-16}$ for all x (machine precision). Hint: you can use the geometric series formulas

$$1/(1-b) \;=\; \sum_{k=0}^{\infty} b^k \;, \quad \text{and} \quad \sum_{k=0}^{n} b^k \;=\; \frac{1-b^{n+1}}{1-b} \;.$$

Also, $\cos(\theta)^k$ is a trigonometric polynomial of degree k (given, if you must, by the $k-$angle formula $\cos(k\theta) = \cos^k(\theta) + \cdots$). Explain your formula $n(a)$ in terms of the *resolution* needed to resolve the narrow peak of f for a close to 1. If $a = 1 - \epsilon$, a reasonable approximation of $f(x)$ near the peak is $f(x) \approx 1/(\epsilon + x^2/2)$ (why?). The maximum of the peak is $f_{\max} = f(0) = 1/\epsilon$. The approximate width of the peak is the x value that gives $f(x) \approx f_{\max}/2$. This is some power of ϵ. What size n do you need before the wavelength of the Fourier mode $e^{2\pi i n x}$ is on the order of the width of the peak? Amazingly, (as I hope you can show) you don't need many more modes than this to get full machine precision on f.

2. (*This scheme asks you to do a number of well known calculations that you easily find online or in books. Please do these calculations without looking in such sources.*) Consider a finite difference approximation to the $1d$ linear advection problem $\partial_t u + s\partial_x u = 0$. The initial value problem is to solve this PDE with initial conditions $u(x.0) = f(x)$. The space and time steps are Δx and Δt respectively. The CFL number is $\lambda = s\Delta t/\Delta x$. The

approximate solution is $u_j^n \approx u(x_j, t_n)$. A $2p+1$ point *one step* scheme has the form

$$u_j^{n+1} = \sum_{|k|\le p} a_k u_{j-k}^n \ . \tag{1}$$

In class we studied the scheme with centered differencing in space and forward Euler in time, which had $p = 1$, $a_{-1} = -\lambda/2$, $a_1 = \lambda/2$, and $a_0 = 1 - \lambda$. This scheme is, alas, unstable. The *order of accuracy* of a scheme of the form (1) is defined as usual. It's the power of Δx in the residual (what you get if you plug in the exact PDE solution into (1)) with λ fixed (or the power ± 1).

(a) The *symbol*, or *multiplier* of the scheme is

$$m(\theta) = \sum_{|k|\le p} a_k e^{-ik\theta} \ .$$

Show that if $|m(\theta)| \le 1$ for all θ, then the scheme is *stable* in l^2 in the sense that

$$\left\| u^{n+1} \right\|_{l^2} \le \left\| u^{n+1} \right\|_{l^2} \ .$$

Here

$$\left\| u^{n+1} \right\|_{l^2}^2 = \sum_j \left(u_j^n \right)^2 \ .$$

If u_j^n is periodic in j, this uses the Plancharel formula of the Discrete Fourier Transform. If u_j^n is defined but not periodic for al j, this is the Phancharel formuls for Fourier series as we did in class.

(b) The *one point first order upwind differencing* scheme uses the approximation $\partial_x u(x,t) \approx (u(x,t) - u(x - \Delta x, t))/\Delta x$, together with forward Euler differencing in time. Show that this method is first order accurate. Find the λ_* so that if $0 < \lambda \le \lambda_*$ then the method is stable. The term *upwind* comes from the idea that the exact solution is being blown by the "wind" to the right, so looking to the left is looking in the "upwind" direction.

(c) The *one point first order downwind* scheme is the same as the upwind scheme except that it uses the downwind approximation $\partial_x u \approx (u(x_\Delta x, t) - u(x,t))/\Delta x$. Show that this method is unstable for any $\lambda > 0$.

(d) The *Lax Wendroff* scheme is the three point one step scheme that is second order accurate. Find the coefficients a_{-1}, a_0, and a_1 of this scheme. Find the *CFL limit* for this scheme, which is λ_* so that the scheme is stable for $\lambda \le \lambda_*$.

(e) The *three point second order upwind scheme* is second order but with a *stencil* of non-zero coefficients consisting of a_0, a_1, and a_2. Find the coefficients that make the scheme second order and find the CFL limit for this scheme.

(f) Prove the easy part (and the important part) of the *Lax equivalence theorem* for this case: if the scheme is accurate of order p and stable, and if the actual solution is smooth enough, then the approximate solution is accurate to order p. This statement is a little vague, so you will have to make it more precise. It is OK to consider the case of periodic initial data.

3. This exercise develops a method to study the Cahn Allen equation. The Cahn Allen equation is a *parabolic* PDE (parabolic means in the heat equation family of PDE, which implies smoothing and the impossibility of running backwards) that models the progress of certain phase change processes. The unknown $u(x,t)$ tells you which phase you are in. $u = \alpha > 0$ is the α phase and $u = \beta < 0$ is the β phase. Since phases mix, it is rare that $u = \alpha$ or $u = \beta$ exactly. Usually $u(x,t)$ is between α and β indicating that the phase at x is not pure. One ingredient in Cahn Allen is a local energy function

$$V(u) \;=\; \left(u^2 - 1\right)^2 \,+\, su \;.$$

For $0 \leq s \leq 1$ (this is a safe range, not the widest possible), $V(u)$ has two local minima, one at $\beta < 0$ and one at $\alpha > 0$. The pure dissipative dynamics that finds one or the other local minimum is

$$\dot{u} \;=\; -V'(u) \;. \tag{2}$$

The set of initial conditions u_0 that "flows" to α is the *basin of attraction* of α, and similarly for β. The separator between these basins is a local maximum of V that is an unstable stationary point for the dynamics (2). If we start with $u(x,0) = u_0(x)$ and run the dynamics (2), then the solution converges as $t \to \infty$ either to α or β for almost every x (unless u_0 is chosen maliciously).

Some physical systems do not have such sharp $\beta \to \alpha$ transitions. The Cahn Allen equation models this by adding some diffusion to (2)

$$\partial_t u \;=\; D \bigtriangleup u \;-\; V'(u) \;. \tag{3}$$

The diffusion term does not allow the $\beta \to \alpha$ transition to be too fast. But making D smaller allows the transition to be more narrow. We are going to solve (3) numerically using a mesh in $2d$ with space size Δx and time step Δt. We will take $\Delta x to 0$ with $\lambda = D\Delta t/\Delta x^2$ fixed.

(a) Suppose we discretize the heat equation part using the standard 5 point Laplace operator and the nonlinear term using forward Euler. Show that the resulting scheme is formally second order accurate as a function of Δx. It is important to keep the CFL ratio λ fixed for this to be true.

(b) Create a second order accurate approximation of (2) using the first order Adams Bashforth (i.e. forward Euler) as a predictor for the second order Adams Moulton (i.e. trapezoid rule) method. The result is explicit but uses an intermediate predicted value that is forgotten at the end of the step. (This is a second order *Runge Kutta* method, as we will see later.)

(c) Create a fourth order approximation to $\partial_x^2 f$ using a centered 5 point scheme. Hint: you can derive the scheme by applying Richardson extrapolation to the second order 3 point approximation. Use this to create a fourth order 9 point approximation to $\triangle u$ in $2d$.

(d) Use the answer to (c) to create a fourth order approximation to the heat equation in $2d$, assuming λ is fixed. Show that the method is stable for small enough λ. Take as a challenge for extra credit if you have time to determine the largest possible stable λ.

(e) Combine parts (b) and (d) and Strang splitting to construct a fourth order method for (3).