**Numerical Methods II**, Courant Institute, Spring 2012
http://www.math.nyu.edu/faculty/goodman/teaching/NumMethII/index.html
**Always** check the class bboard on the blackboard site from home.nyu.edu (click on academics, then on Numerical Methods II) before doing any work on the assignment.

# Assignment 6

**Corrections:** (none yet)

1. Consider the linear hyperbolic system $\partial_t u + A\partial_x u = 0$ where $u$ has $n$ components and $A$ is an $n \times n$ matrix with $n$ real and distinct eigenvalues.
   (a) Assuming $u(x,t)$ is a smooth function of $x$ and $t$, write the $u(x, t+\Delta t)$ in terms of $u(x,t)$ and time derivatives up to second order that is accurate up to order $\Delta t^2$.
   (b) Assume that $u$ satisfies the PDE and express $\partial_t u$ and $\partial_t^2 u$ in terms of $\partial_x u$ and $\partial_x^2 u$ (and $A$ and $A^2$).
   (c) Suppose we have $\Delta x$ and $\Delta t$ and $k_k = k\Delta x$ and $t_j = j\Delta t$ as usual. Write an approximation of $u(x_k, t_{j+1})$ using the approximations of part (a) and (b) and the second order centered difference formulas for $\partial_x u$ and $\partial_x^2 u$. The resulting approximation should use $u(x_{k-1}, t_j)$, $u(x_k, t_j)$, and $u(x_{k+1}, t_j)$.
   (d) Use this to create a finite difference approximation for the PDE that gives $u_k^{j+1}$ in terms of $u_{k+1}^j$, $u_k^j$, and $u_{k-1}^j$. Show that if $\Delta t = \lambda \Delta x$ then the resulting method is second order accurate. This is the *Lax Wendroff* method.
   (e) Use von Neumann analysis to show that the Lax Wendroff method is stable if $\Delta t \leq C\Delta x$ and find a formula for the largest possible value of $C$ in terms of the eigenvalues of $A$. (Lax and Wendroff would not have proposed an unstable method.)

2. Consider (for simplicity only) the advection equation $\partial_t u + \partial_x u = 0$ and consider the time stepping method that uses backward Euler differencing in time and centered second order differencing in space. Show that this method is stable and first order accurate if $\Delta t = \lambda \Delta x$ with no constraint on $\lambda$. At each time step you have to solve a system of linear equations. How does the condition number of this system depend on $\Delta t$ (e.g. proportional to some power of $\Delta t$)?

3. Consider the second order centered difference approximation to $\triangle u = f$ in the unit square in two dimensions with Dirichlet boundary conditions.
   (a) Show that the eigenvectors of the resulting matrix are products of sines: $u_{jk} = \sin(k_x x_j, k_y y_k)$ and find a formula for the corresponding eigenvalue.

(b) *Without doing trigonometric sums* give a proof that distinct eigenfunctions are orthogonal in $l^2$.

(c) What is the condition number of the discrete Laplace operator as a function of $\Delta x$.

4. Suppose we use the conjugate gradient algorithm without preconditioning to solve the discrete equation system from question 3 with $f(x, y) = xy$ and initial guess $u_{0,jk} = 1$ if $j = 1, k = 1$ and $u_{0,jk} = 0$ otherwise. Show that the error after $n/2$ steps does not go to zero as $n \to \infty$ with $\Delta x = 1/n$. Is this consistent with the known convergence estimates for the conjugate gradient method and the condition number of the problem?