(2) Pf: let $y_n = x(t_n)$, want to show

$\|y_n - x_n\| \le \Delta t\, C(t_n)$.

Consistency + Stability argument.

Consistency: let $y_n = x(t_n)$, then y_n is an approximate solution of the discrete equation

$$y_{n+1} = y_n + \Delta t\, f(y_n) + \Delta t\, R_n$$

where

(*) $$\|R_n\| \le \Delta t \cdot A(t_n) \qquad A(t) \text{ indep of } \Delta t. \text{ if } |\Delta t| \le 1$$

Pf of (*):

$$x(t_{n+1}) = x(t_n) + \Delta t\, f(x(t_n)) + \Delta t\, R_n$$

Taylor

$$x_j(t_{n+1}) = x_j(t_n) + \dot{x}_j(t_n)\Delta t + \frac{1}{2}\ddot{x}_j(\xi_j)\Delta t^2$$

$j = 1, \dots, d$

(3)

with $t_n \le \xi_j \le t_{n+1}$.

(each component possibly a different ξ_j).

But $\ddot{x}(t) = f'(x(t))\, f(x_t)$

so $\|\ddot{x}_j\| \le M \cdot M \cdot \underbrace{\max_{t \in (t_n, t_{n+1})} x(t)}_{B(t) \text{ indep of } \Delta t}$

so $\|R_n\| \le d\,B(t_n) = A(t_n) \cdot \Delta t$.

Stability:

if $x_{n+1} = x_n + \Delta t\, f(x_n)$ & and $x_0 = y_0$

$y_{n+1} = y_n + \Delta t\, f(y_n) + \Delta t\, R_n$

with bounds on f, f', ~~and~~, and R_n

as above, then

(*) $\|x_n - y_n\| \le D(t_n) \max \|R_n\|$

(4)

pf of (★), by induction on n

~~$x_{n+1} - x_1$~~

$$x_{n+1} - y_{n+1} = x_n - y_n + \Delta t\left(f(x_n) - f(y_n)\right) - \Delta t\, R_n$$

$$\|f(x_n) - f(y_n)\| \le M \cdot \|x_n - y_n\|, \text{ so}$$

$$\|x_{n+1} - y_{n+1}\| \le (1 + M\Delta t)\|x_n - y_n\| + \Delta t\, \|R_n\|$$

$$D(t) = t e^{Mt}$$

$$D(t + \Delta t) \le (1 + M\Delta t) D(t) + \Delta t$$

QED

~~General~~ philosophy

error $\sim \Delta t\, C(t)$ is 1^{st} order = not very accurate.

8

~~Lemma of~~

Consequence ~~of~~ polynomials as an approximation space.

Lemma 2: If $u(t) \in C^{p+1}$

$$|\partial_t^{\alpha} u(t)| \leq M \quad \text{all } \alpha \leq p+1$$

~~then~~ and ~~K~~ $I = [-K\Delta t, K\Delta t]$ is given, then there is a polynomial $v(t)$ of degree p so that

$$|v(t) - u(t)| \leq C_{K,p} \, \Delta t^{p+1}$$

when $\cancel{|t| \leq K\Delta t}$ $t \in I$

stability of interpolation

Lemma 3: Let $v(t)$ and $w(t)$ be polynomials of degree p. Let $t_0, \ldots, t_p$ ~~$\in [-L\Delta t, L\Delta t]$ with $|t_j - t_k| \geq$~~

with $t_j \in I$ and $|t_i - t_j| \geq \alpha \Delta t$ whenever $i \neq j$. Then

$$\max_{t \in I} |v(t) - w(t)| \leq C_{K,\alpha,p} \cdot \max | v(t_i) - w(t_i) |$$

~~Pf of lemma 4 from 2 + 3,~~

Lemma 4 If $|\partial_x^\alpha f| \le M$ all $|\alpha| \le p+1$ and $\dot{x} = f(x)$ then

$$|\partial_t^\alpha x(t)| \le C_{d,p} M^{p+1} \quad \text{for } \alpha \le p+1$$

$$|\partial_t^\alpha f(x(t))| \le C_{d,p} M^{p+1}$$

Pf $\dot{x} = f \Rightarrow \cancel{|\partial_t f|}$ $(\cancel{\alpha}\ p=0)$

$$|\partial_t x| \le M.$$

$$\ddot{x} = \partial_t \dot{x} = \partial_t f(x) = f'(x)\dot{x}$$

$$\le |f'(x) \cdot f(x)|$$

$$\le C_{d,2} M^2$$

$$\dddot{x} = \partial_t \left(f'(x) f(x) \right)$$

$$= f''(x) f(x) \cdot f(x) + f'(x) \cdot f'(x)$$

$$|\dddot{x}| \le C_d M^3$$

etc

(10)

Pf of Lemma 1 from 2, 3, ~~4~~.

From L3, $x(t)$ satisfies

$$|\partial_t^\alpha f(x(t))| \le M, \quad \cancel{|\partial_t f(x(t))| \le M \text{ also}}.$$

Therefore (L2~~4~~) there is a polynomial $u_n(t)$

~~with $h(t)$ with~~

degree p with $|u_n(t) - f(x(t))|$

$\le C\, \Delta t^{p+1}$ when $|t - t_n| \le K\Delta t$.

~~By L3 the interpolat.~~

Let $h_n(t)$ = the interpolating polynomial.

L3 says $|h_n(t) - u_n(t)| \le$

$$C \cdot \max_{i \le p} |h_n(t_{n-i}) - u_n(t_{n-i})|$$

$$= C \cdot \max_{i \le p} |f(x(t_{n-i})) - u_n(t_{n-i})|$$

$$= C \cdot \max |f(x(t)) - u_n(t)|$$

$$= C \cdot \Delta t^{p+1}.$$

Then $\cancel{\displaystyle\int_{t_n}^{t_{n+1}} |f(x(t)) \, dt - u_n(t)| \, dt}$

$$\int_{t_n}^{t_{n+1}} |f(x(t)) - h_n(t)| \, dt$$

$$\leq \int |f(x(t)) - u_n(t)| \, dt \quad \leftarrow L_2$$

$$+ \int |u_n(t) - h_n(t)| \, dt \quad \cancel{\text{bn}}\ L_3$$

$$\leq \Delta t \cdot \left(C \Delta t^p + c \Delta t^p \right)$$

QED

Lemma (Stability)

If $\quad y_{n+1} = y_n + \Delta t \sum_{i=0}^{p} \beta_i f(y_{n-i}) + \Delta t R_n$

$$x_{n+1} = x_n + \Delta t \sum \beta_i f(x_{n-i})$$

$x_0 = y_0, \ldots, x_p = y_p$

then $|x_n - y_n| \leq C(t_n, p) \max_{t_k \leq t_n} |R_k|$

(12)

Pf set $X_n = (x_n, x_{n-1}, \cdots, x_{n-p})$

$Y_n = (y_n, \cdots, y_{n-p})$

$R_n = (r_n, \cdots\ r_{n-p})$

$\|X\| = \max(|x_n|, \cdots |x_{n-p}|)$

$F(X) = (\cancel{f(x_n)}, x_n, \cdots\ x_{n-p+1})$

Then $(\sum \beta_i f(n-i), \cancel{x_n, \cdots x_{n-p}}\ 0 \cdots 0)$

$X_{n+1} = X_n + \Delta t\ F X$

$A X = (x_n, x_n, \cdots\ x_{n-p+1})$

has $\|A X\| \le \|X\|$.

$\|F(X) - F(Y)\| \le M. \|X - Y\|$.

~~Now~~

$\|X_{n+1} - Y_{n+1}\| \le (1 + M \Delta t) \| X_n \cancel{+} - Y_n\|$

$+ \Delta t \|R_n\|$.