① Numerical Methods II goodman@cims.nyu.edu
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Higher order Adams Bashforth etc.

To make a higher order method

- say something more nearly true about the exact solution
- let it be stable

For Adams methods, start with

$$x(t+\Delta t) = x(t) + \int_t^{t+\Delta t} f(x(s))\,ds$$

replace $f(x(t))$ with a degree $p-1$ interpolating polynomial

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} g_n(s)\,ds$$

where

(i) $g_n(s)$ = a polynomial degree $p-1$

(ii) $g_n(t_{n-k}) = f(x_{n-k})$ for $k=0, \ldots, p$

(2)

e.g. $p = 1$: g_n = degree 0 =

$g_n = f(x_n)$ - get forward Euler

e.g. $p = 2$

$$g_n(s) = f(x_n) + (s - t_n) \cdot \frac{f(x_n) - f(x_{n-1})}{t_n - t_{n-1}}$$

(Newton form of interpolating polynomial)

$$\int_{t_n}^{t_{n+1}} g_n(s)\,ds = \Delta t_n\, f(x_n)$$

$$+ \frac{\Delta t_n^{\cancel{2}}}{2} \frac{f(x_n) - f(x_{n-1})}{\cancel{\Delta t_n}}$$

$$= \Delta t_n \left(\frac{3}{2} f(x_n) - \frac{1}{2} f(x_{n-1}) \right)$$

$$x_{n+1} = x_n + \Delta t \left(\frac{3}{2} f(x_n) - \frac{1}{2} f(x_{n-1}) \right)$$

2nd order Adams Bashforth method.

general form of A-B methods

$$x_{n+1} = x_n + \Delta t \cdot \sum_{i=0}^{p-1} \beta_i f(x_{n-i})$$

<u>Important remark</u> $\Delta t_n = t_{n+1} - t_n$

may be non-uniform. Good thing about Adams methods.

(3)

Remark: It's hard to get started if you know x_0, not x_1. Will not have an exact value of x_1.

<u>Theorem</u>: (i) $|\partial_x^\alpha f(x)| \le M$ all x, $|\alpha| \le p$

(ii) $\gamma \le \Delta t_{n+1}/\Delta t_n \le \delta$ all n, $\alpha \ge 0$

(iii) $\Delta t_n \le h$ all n

(iv) $|x_k - x(t_k)| \le C_0 h^p$ for $0 \le k < p$

Then $|x_k - x(t_k)| \le C(t_k)\, h^p$

<u>Pf</u> For constant $\Delta t = h$, set

$$x_{n+1} = x_n + \Delta t \sum_{i=0}^{p-1} \beta_{i,n} f(x_{n-i})$$

Lemma 1 (shape regularity): The shape regularity condition (ii) implies that

$$|\beta_{i,n}| \le C_{\gamma,\delta,p}$$

for all i, n.

④

Proof of Lemma 1:

Lagrange interpolation formula: of $h(t)$ is a polynomial degree $p-1$ with $h(t_i) = f_i$ for $i = 0, \dots, p-1$, then

$$h(t) = \sum L_i(t) f_i$$

where $L_i(t)$ are the Lagrange pt interpolating polynomials

$$L_i(t) = \prod_{j \neq i} \frac{t - t_j}{t_i - t_j}.$$

Under the hypotheses, $|t - t_j| \leq c_1 h$ and $|t_i - t_j| \geq c_2 h$ where $h = \min_{\text{relevant } i} |t_i - t_{i+1}|$.

~~So~~ Then

$$\int_{t_n}^{t_{n+1}} h(t)\,dt = \sum \underbrace{\left(\int_{t_n}^{t_{n+1}} L_i(t)\,dt \right)}_{\Delta t \beta_i} f_i$$

QED (Lemma 1).

(5)

Lemma 2 (consistency): Under hypotheses (i) and (ii), if $y_n = x(t_n)$ then

$$y_{n+1} = y_n + \Delta t \sum_{i=0}^{p-1} \beta_{i,n} f(y_{n-i}) + \Delta t\, r_n$$

where

$$|r_n| \le C \cdot \Delta t_n^p .$$

Proof: Step 1: $f(x(t))$ is ~~[illegible]~~ smooth.

Step 2: smooth $f \Rightarrow$ small residual

Step 1: $\frac{d}{dt} f(x(t)) = f'(x) \cdot \dot{x} = f'(x) f(x) =$ bounded

$$\frac{d}{dt}\frac{df}{dt} = \frac{d}{dt}\left(f'(x) f(x)\right) = f''(x) f(x) f(x) + f'(x) f'(x) f(x)$$

etc.

Step 2: Let $h_n(t)$ be the Taylor expansion of