(1)

Implicit methods

For now write $h = \Delta t$ and suppose $t_n = nh$ with a fixed time step. (ok for some applications, bad for others).

<u>Implicit method</u> = have $x_n, x_{n-1}, \ldots,$ $f(x_n), f(x_{n-1}), \ldots$. Get x_{n+1} by solving an equation involving x_{n+1}.

Why? It can be more stable, compact. (see below for lots on stability).

<u>eg</u> Adams Moulton:

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} p_n(t)\, dt \quad \text{where}$$

p_n interpolates $f(x_{n+1}), f(x_n), \ldots f(x_{n-s})$

Accuracy $= h^{s+1}$ — one better than Adams-Bashforth

(2)

eg. <u>Trapezoid rule</u> $\frac{x_{n+1} - x_n}{h} = \frac{f(x_{n+1}) + f(x_n)}{2}$

2^{nd} order because it is <u>centered</u> about $t_{n+\frac{1}{2}} = t_n + \frac{1}{2}h$.

eg Backward differentiation formula, BDF.

$$\dot{x}(t_{n+1}) = \frac{1}{h}\left(x(t_{n+1}) - x(t_n)\right) + O(h)$$

1^{st} order

$$\dot{x}(t_{n+1}) = \frac{1}{h}\cdot\left(\frac{3}{2}x(t_{n+1}) - 2x(t_n) + \frac{1}{2}x(t_{n-1})\right) + O(h^2)$$

2^{nd} order.

Methods have that order

$$\frac{x_{n+1} - x_n}{h} = f(x_{n+1})$$ – Backward Euler

$$\frac{3}{2}x_{n+1} - 2x_n + \frac{1}{2}x_{n-1} = h f(x_{n+1})$$ – 2^{nd} order BDF.

③

Other methods: Nystrom

$$x_{n+1} = x_{n-1} + \int_{t_{n-1}}^{t_{n+1}} p_n(t)\, dt$$

e.g. Mid point = leap frog

$$x_{n+1} = x_{n-1} + 2h\, f(x_n)$$

2^{nd} order because of centering.

(4)

Linear multistep methods LMM

explicit

$$x_{n+1} = a_0 x_n + \dots + a_k x_{n-k}$$

(explicit) $+ \Delta t \left(b_0 f(x_n) + \dots + b_k f(x_{n-k}) \right)$

(implicit) $\Delta t \left(b_{-1} f(x_{n+1}) + \dots + b_k f(x_{n-k}) \right).$

stability: zero stability.

~~The~~ Works for $\dot{x} = 0$ $(f(x) = 0)$.

solution of linear recurrence.

$$z^{k+1} = a_0 z^k + \dots + a_k$$

– stability polynomial

Note: if consistent, $z = 1$ is a root.

Other roots either inside or simple on unit circle. – Lie-power's thm on stability.