(1) ~~4~~ 43 – Discrete Fourier Transform – DFT

Algebra of the DFT in one D.

f_j is periodic with period n if $f_{j+n} = f_j$

The set of periodic discrete functions in ~~n-dimensional~~ n dimensions. $\mathbb{R}^n$ or $\mathbb{C}^n$

inner product $\langle f, g \rangle_{\ell^2} = \frac{1}{n} \sum f_j \bar{g}_j$

Discrete Fourier modes = complex exponentials

$u_\alpha \in \mathbb{C}^n \qquad u_{\alpha,j} = e^{2\pi i \alpha j/n}$

Lemma: The vectors u_α for $\alpha = 0, \dots, n-1$ are an orthonormal basis of $\mathbb{C}^n$.

Pf: We have n vectors. They are ~~an orthonormal~~ a basis if they are orthonormal if $0 \le \alpha < \beta \le n-1$ then

$$\langle u_\alpha, u_\beta \rangle = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

②

$$\langle u_\alpha, u_\beta \rangle_{\ell^2} = \frac{1}{n} \sum_{j=0}^{n-1} \exp\left(\frac{2\pi i (\beta-\alpha) j}{n}\right)$$

if $\alpha = \beta$, this is $\frac{1}{n} \sum_{j=0}^{n-1} 1 = 1$

if $\alpha \neq \beta$ and $\beta - \alpha = \gamma$ with $1 \le \gamma \le n-1$

$$\sum_{j=0}^{n-1} \exp\left(\frac{2\pi i \gamma}{n} \cdot j\right) = \sum_{j=0}^{n-1} z^j = \frac{1-z^n}{1-z}$$

with $z = \exp\left(\frac{2\pi i \gamma}{n}\right)$ and $\underline{z^n = 1}$

but $z \neq 1$ (because $\gamma > 0$ and $\gamma < n$)

so $\sum z^j = \frac{\text{zero}}{\text{not zero}} = 0.$

<u>Fourier coefficients</u> $\hat{f}_\alpha = \langle u_\alpha, f \rangle_{\ell^2}$

$$= \frac{1}{n} \sum e^{-\frac{2\pi i \alpha j}{n}} \cdot f_j$$

Fourier representation $f = \sum \hat{f}_\alpha u_\alpha$

$$f_j = \sum_{\alpha=0}^{n-1} \hat{f}_\alpha e^{2\pi i \alpha j / n}.$$

Fourier inversion formula

$$\langle f, f \rangle_{\ell^2} = \sum_{\alpha=0}^{n-1} |\hat{f}_\alpha|^2 \quad \text{Plancherel thm.}$$

③

Multi-Dimensional

f_{jk} period n in the x-direction
period m in the y-direction

$f_{j+n,k} = f_{j,k}$, $f_{j,k+m} = f_{j,k}$

Fourier basis

$$u_{\alpha\beta jk} = \exp\left(\frac{2\pi i\alpha j}{n}\right)\exp\left(\frac{2\pi i\beta k}{m}\right)$$

$$= \exp\left(2\pi i\left(\frac{\alpha j}{n} + \frac{\beta k}{m}\right)\right)$$

$$\langle f, g\rangle = \frac{1}{nm}\sum_{j,k} f_{jk}\, g_{jk} \quad \text{etc.}$$

Aliasing as a function of α,

$$\hat{f}_\alpha = \frac{1}{n}\sum e^{\frac{2\pi i\alpha j}{n}} f_j$$

is a periodic fn of α with period n

$$\hat{f}_{\alpha+n} = \frac{1}{n}\sum e^{\frac{2\pi i(\alpha+n)j}{n}} f_j =$$

alias = different name for the same thing

So any n consecutive u_α are an ortho-normal basis

4

For example, $n = 2m+1 =$ odd

$n = -m, \cdots, 0, \cdots, m$ = symmetric about zero.

<u>Relation to Fourier series.</u>

suppose $f(x+1) = f(x)$ – periodic period 1.

Fourier modes $u_\alpha(x) = e^{2\pi i \alpha x}$

$$\hat{f}_\alpha = \langle u_\alpha, f \rangle = \int_0^1 e^{-2\pi i \alpha x} f(x)\, dx$$

$$\langle u_\alpha, u_\beta \rangle = \delta_{\alpha\beta} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ 1 & \text{if } \alpha = \beta \end{cases}$$

for any (integers) α, β

$$f = \sum_{\alpha=-\infty}^{\infty} \hat{f}_\alpha u_\alpha$$

$$f(x) = \sum_{\alpha=-\infty}^{\infty} \hat{f}_\alpha e^{2\pi i \alpha x}$$

Fourier inversion formula

$$\int_0^1 |f(x)|^2 dx = \sum_{-\infty}^{\infty} |\hat{f}_\alpha|^2$$

Plancherel formula

Smoothness and Decay: The Fourier representation is efficient for smooth functions.

Efficient: a cheap approximation - a small number of terms - can be very accurate.

Reason: rapid decay of Fourier coefficients

Two versions: smooth & analytic

(i) smooth: $|\partial_x^p f(x)| \le A_p$ all x

(ii) analytic: $|f(x+iy)| \le A$ all $|y| \le L$

(i) ~~for~~ smooth: first observe that

$$|\hat{f}_\alpha| = \left| \int_0^1 e^{-2\pi i \alpha x} f(x)\,dx \right|$$

$$\le \int_0^1 |f(x)|\,dx \le A_0$$

A bounded function has bounded Fourier coefficients. To continue, use

$$\partial_x e^{-2\pi i \alpha x} = -2\pi i \alpha\, e^{-2\pi i \alpha x}$$

so $\dfrac{-1}{2\pi i\alpha}\partial_x e^{-2\pi i\alpha x} = e^{-2\pi i\alpha x}$

and integrate by parts, using periodic boundary conditions:

$$\hat{f}_\alpha = -\frac{1}{2\pi i\alpha}\int\left(\partial_x e^{-2\pi i\alpha x}\right) f(x)\,dx$$

$$= \frac{1}{2\pi i\alpha}\int e^{-2\pi i\alpha x}\left(\partial_x f(x)\right)dx$$

so

$$\left|\hat{f}_\alpha\right| \le \frac{1}{2\pi|\alpha|}A_1$$

continuing:

$$* \qquad \left|\hat{f}_\alpha\right| \le \frac{1}{(2\pi)^p}\cdot A_p\cdot\frac{1}{|\alpha|^p}$$

slightly simpler form: the function e^{ikx} is a complex exponential with wave number k.

$e^{2\pi i\alpha x}$ has wave number $k_\alpha = 2\pi\alpha$.

$*$ is $\left|\hat{f}_\alpha\right| \le \dfrac{1}{|k_\alpha|^p}A_p$.

(7)

<u>Bottom line</u>: if f is a smooth function, the Fourier coefficients decay faster than any power of α.

(ii) analytic: move the contour

$$\hat{f}_\alpha = \int_0^1 e^{-2\pi i \alpha x} f(x)\,dx = \int_0^1 e^{-2\pi i \alpha (x+iy)} f(x+iy)\,dx$$

If $\alpha > 0$, take $y = -L$ & get

$$|\hat{f}_\alpha| \le e^{-2\pi\alpha L} \cdot A = e^{-|k_\alpha| L} A$$

(*) $$|f_\alpha| \le e^{-2\pi |\alpha|} A = e^{-|k_\alpha| L} A$$

The Fourier coefficients of an analytic decay exponentially with α.

<u>Efficient representation lemma</u>

<u>lemma</u> $\displaystyle\sum_{\alpha > m} \frac{1}{\alpha^p} \cong \int_m^\infty \frac{1}{\alpha^p}\,d\alpha = \frac{1}{p-1} m^{p-1}$

if $p > 1$

$$\left| f(x) - \sum_{|\alpha| \le m} \hat{f}_\alpha e^{2\pi i \alpha x} \right| \le \sum_{|\alpha| > m} \hat{f}_\alpha e^{2\pi i \alpha x}$$

8

$$|\hat{f}(x) - f^m(x)| \le \frac{C_p}{m^{p-1}} \quad \text{for any } p$$

Similarly $\sum_{\alpha = m}^{\infty} e^{-2\pi \alpha L} \le \dfrac{e^{-2\pi L m}}{1 - e^{-2\pi L}}$

so $|f(x) - \hat{f}^m(x)| \le C \; e^{-2\pi L m}$

Spectral accuracy

error in m-term (or $2m+1$-term) approx

$$\le \begin{cases} \text{any power of } m & \text{- smooth fn} \\ \text{an exponential in } m & \text{- analytic.} \end{cases}$$

Discrete & continuous Fourier representation

$f(x)$ = smooth fn

f^n = sampled fn (not Fourier approx)

$f^n_j = f\left(j/n\right) = f(x_j) \qquad x_j = j \cdot h, \quad h = \frac{1}{n}$

$$\widehat{f^n_\alpha} = \frac{1}{n} \sum_j e^{-2\pi i \alpha j/n} f^n_j = h \sum_j e^{-2\pi i \alpha x_j} f(x_j)$$

$$\cong \int e^{-2\pi i \alpha x} f(x)\, dx = \hat{f}_\alpha$$

Relationship between $\hat{f}^n_\alpha$ - discrete Fourier transform and $\hat{f}_\alpha$ - continuous Fourier integral:

Aliasing formula

$$f_j = f(x_j) = \sum_{\alpha=-\infty}^{\infty} \hat{f}_\alpha e^{2\pi i \alpha x_j}.$$

$$\hat{f}^n_\alpha = \sum_{\alpha=-\infty}^{\infty} \underbrace{\left(\frac{1}{n} \sum_{j=0}^{n-1} e^{-2\pi i (\beta-\alpha) j/n} \right)}_{= \begin{cases} 1 & \text{if } \beta = \alpha + kn \text{ some int. } k \\ 0 & \text{otherwise} \end{cases}} \hat{f}_\beta$$

(*) $$\hat{f}^n_\alpha = \sum_{k=-\infty}^{\infty} \hat{f}_{\alpha+kn}$$

corollary: $$\hat{f}^n_\alpha - \hat{f}_\alpha = \sum_{k \neq 0} \hat{f}_{\alpha+kn}$$

if $|\alpha| \leq m$, get

$$\left| \hat{f}^n_\alpha - \hat{f}_\alpha \right| < \begin{cases} c_p m^{-p} & \text{any } p \\ A e^{-cm} & \end{cases}$$

DFT $\cong$ continuous Fourier series for smooth function

(10)

corollary $|\hat{f}^n_\alpha| \le \begin{cases} C_p |\alpha|^{-p} \\ C_1 e^{-C_2 |\alpha|} \end{cases}$

rapid decay of DFT coefficients.

corollary: take $\alpha = 0$

$$\left| h \sum_{0 \le x_j < 1} f(x_j) - \int_0^1 f(x)\,dx \right|$$

$$\le \begin{cases} C_p \; h^p \\ C_1 e^{-C_2/h} \end{cases}$$

For smooth functions, uniform spacing trapezoid rule numerical integration is spectrally accurate.

corollary: Fourier interpolation / trigonometric interp.

given $f = (f_0, \ldots, f_{n-1})$ interpolate

$$\tilde{f}(x) = \sum_{|\alpha| \le m} \hat{f}_\alpha \, e^{2\pi i \alpha x}$$

interpolation condition: $\tilde{f}(x_j) = f_j$

(11)

Spectral accuracy. if $f_j = f(x_j)$

then $|\tilde{f}(x) - f(x)| \le \begin{cases} C_m\, m^{-p} \\ C_1\, e^{-C_2 m} \end{cases}$

General rule If the solution to a differential equation is smooth and periodic, you get spectral accuracy using a discrete Fourier representation

Spectral methods.

~~Other boundary conditions~~

~~Dirichlet: $f(0) = 0$, $f(1) = 0$~~

~~1^{st} extend f to be odd in $(-1, 0)$~~

~~for $0 \le x \le 1$ set $f(-x) = -f(x)$~~

~~2^{nd} extend f to all x to be periodic, period 2~~

~~$f(x + 2n) = f(x)$ for $-1 \le x \le 1$~~

(12)

Application Fourier modes (discrete and continuous) are eigenfunctions / eigenvectors of important operators.

Eigenfunctions depend on boundary conditions.

Trick: enforce boundary conditions using symmetry. Dirichlet $\Leftrightarrow$ odd, Neumann $\Leftrightarrow$ even

Dirichlet suppose $f(0)=0$, $f(L)=0$.

step 1) define $f(-x)=-f(x)$ for $0\le x\le L$

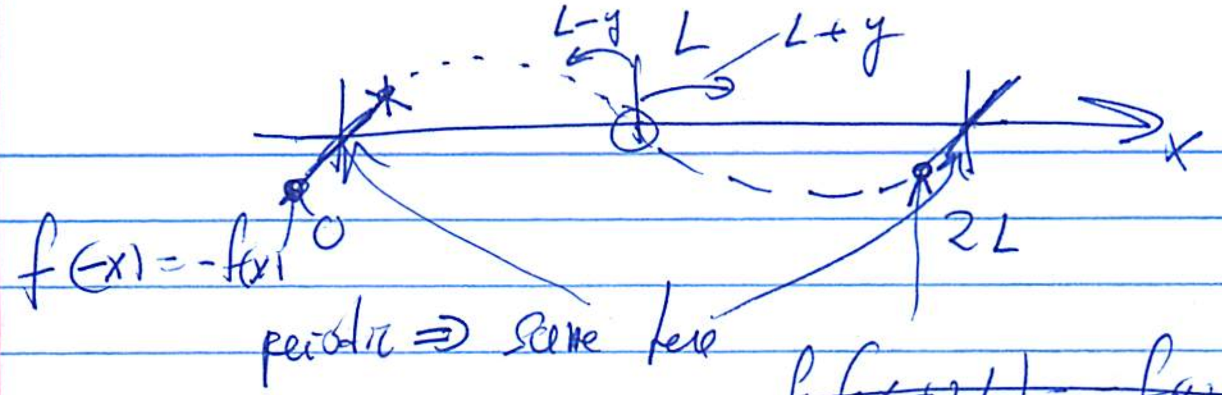
step 2) define $f(x+2Ln)=f(x)$ for

$$-L\le x\le L, \quad n \text{ an integer}$$

Result: f is periodic period $2L$ and odd.

Verification if $f(x)$ is continuous, period $2L$ and odd, then $f(0)=0$, $f(L)=0$.

Pf $f(0)=\lim_{\varepsilon\to 0} f(\varepsilon)=\lim_{\varepsilon\to 0} -f(-\varepsilon)=0.$



~~f(x+~~

$$\cancel{f(x+2L) = f(x)}$$

$$f(-x+2L) = f(-x) = -f(x)$$

set $x = L-y$, get

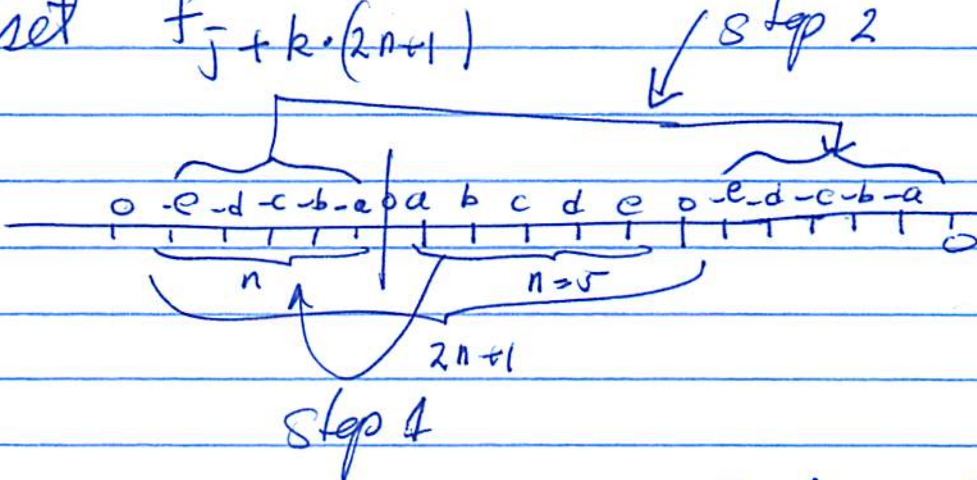
$$f(-\cancel{L}+y+\cancel{2}L) = -f(L-y)$$

$$f(L+y) = -f(L-y) \Rightarrow f(L) = 0 \text{ too.}$$

<u>Discrete</u> $f_0 = 0$, $f_{n+1} = 0$

step (1) for $j = 1, \dots, n$, set $f_{-j} = -f_j$

step (2) set $f_{j+k\cdot(2n+1)}$



get f periodic period $2n+1$ with $f_{-j} = -f_j$
and $f_{n+1} = 0$

(14)

Fourier series period L - choice of normalization.

- normalization by orthonormal eigenfunction

$$u_\alpha(x) = \frac{1}{\sqrt{L}} e^{2\pi i \alpha x/L}$$

$$\langle u_\alpha, u_\beta \rangle = \int_0^L \overline{u}_\alpha(x)\, u_\beta(x)\, dx$$

$$= \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

~~normalization by representation/inversion form~~

$$f = \sum f_\alpha u_\alpha$$

$$\langle u_\beta, f \rangle = \sum \hat{f}_\alpha \langle u_\beta, u_\alpha \rangle$$

$$= \hat{f}_\beta$$

$$\hat{f}_\beta = \frac{1}{\sqrt{L}} \int_0^L e^{-2\pi i \beta x/L} f(x)\, dx$$

$$f(x) = \frac{1}{\sqrt{L}} \sum \hat{f}_\alpha e^{2\pi i \alpha x/L}$$

- normalization by representation/inversion formula

$$f(x) = \sum_\alpha \hat{f}_\alpha e^{2\pi i \alpha x/L}$$

15

$$\int_0^L e^{-2\pi i \beta x/L} f(x)\,dx = \sum_\alpha \hat{f}_\alpha \int_0^L e^{2\pi i(\alpha-\beta)x/L}\,dx$$

$$= \hat{f}_\alpha \cdot \begin{cases} L & \text{if } \alpha=\beta \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{f}_\alpha = \frac{1}{L}\int_0^L e^{-2\pi i \alpha x/L} f(x)\,dx.$$

In particular, $f(x+2)=f(x)$ has

$$f(x) = \sum_\alpha \hat{f}_\alpha e^{\pi i \alpha x}$$

with

$$\hat{f}_\alpha = \frac{1}{2}\int_{-1}^{1} e^{-i\pi\alpha x} f(x)\,dx$$

if f is odd then

$$\hat{f}_\alpha = \frac{1}{2}\int_0^1 \left(e^{-i\pi\alpha x} \cancel{+} - e^{i\pi\alpha x}\right) f(x)\,dx$$

$$= \frac{\cancel{-a}\, i}{-2i} \int_0^1 \frac{\cancel{\$}\, e^{i\pi\alpha x} - e^{-i\pi\alpha x}}{2i} f(x)\,dx$$

$$\hat{f}_\alpha = -i \int_0^1 \sin(\pi\alpha x)\, f(x)\,dx$$

16

note $\hat{f}_{-\alpha} = -\hat{f}_\alpha$, ~~so~~ $\hat{f}_0 = 0$, so

$$f(x) = \sum_{\alpha=1}^{\infty} \hat{f}_\alpha \left(e^{\pi i \alpha x} - e^{-\pi i \alpha x}\right)$$

$$= 2i \sum_{\alpha=1}^{\infty} \hat{f}_\alpha \sin(\pi \alpha x)$$

Fourier sine series:

$$\tilde{f}_\alpha = \int_0^1 \sin(\pi \alpha x)\, f(x)\, dx$$

$$\hat{f}_\alpha = -i\, \tilde{f}_\alpha$$

$$f(x) = 2 \sum_{\alpha=1}^{\infty} \tilde{f}_\alpha \sin(\pi \alpha x)$$

(check: $f(x) = \sin(\pi x) \Rightarrow \tilde{f}_1 = \int_0^1 \sin^2(\pi x)\, dx = \frac{1}{2}$

$$\Rightarrow \quad 2 \sum_\alpha \tilde{f}_\alpha \sin(\pi \alpha x) = \sin(\pi x) \quad \checkmark \;).$$