(1)

Time Dependent PDE

Heat equation $\partial_t u = D \, \Delta u$

Wave eqn $\partial_t^2 u = c^2 \, \Delta u$

1 d linear advection $\partial_t u + s \, \partial_x u = 0$

Initial value problem: give $u(x,0)$ & compute $u(x,t)$ for $t \geq 0$

Wave eqn: give $u_0(x) = u(x,0)$, $v_0(x) = \partial_t u(x,0)$

(For now) Ignore boundary conditions.

Representative solutions.

→ Heat eqn $u(x,t) = \frac{1}{(4\pi D t)^{d/2}} e^{-\frac{|x|^2}{4Dt}}$

concentrated for small t, broader & smoother for large t.

Larger $D \Rightarrow$ faster spreading.

→ Wave eqn - 1 dim $\partial_t^2 u = c^2 \partial_x^2 u$

(2)

$$u(x,t) = f(x+ct) + g(x-ct)$$

left moving wave   right moving

$$\text{speed} = c = \sqrt{c^2}$$

Every soln has this form: fit initial data

$t=0$: $u_0(x) = f(x) + g(x)$

$$v_0(x) = \partial_t u(x,0) = c\left(f'(x) - g'(x)\right)$$

$$\frac{u_0'(x) + \frac{1}{c} v_0(x)}{2} = f'(x)$$

$$\frac{u_0'(x) - \frac{1}{c} v_0(x)}{2} = g'(x)$$

<u>Wave eqn 3-D</u>

$$u(x,t) = \frac{1}{t}\,\delta(|x| - ct)$$

spherically expanding sharp wave

amplitude decreases as the wave spreads.

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Advection (model problem)

$$u(x,t) = f(x-st) \qquad s = \text{speed.}$$

Numerical soln w/o boundaries

$j \in \mathbb{Z}^d = (j_1, \dots, j_d) = d$ integers

Δx = spatial grid size

$$x_j = \Delta x\, j = (\Delta x\, j_1, \dots, \Delta x\, j_d)$$

Δt = time step. $\qquad t_n = n\Delta t$

$$u_j^n \cong u(x_j, t_n)$$

discrete Laplace op ~~$(\Delta_h v)_j =$~~

$$(\Delta_h v)_j = \frac{1}{\Delta x^2} \sum_{\alpha=1}^{d} \left(v_{j+e_\alpha} + v_{j-e_\alpha} - 2v_j \right)$$

$$e_\alpha = (0, \dots, 1, \dots, 0)$$

slot α.

Explicit schemes

Heat eqn $\quad \dfrac{u_j^{n+1} - u_j^n}{\Delta t} = D\, \Delta_h u_j^n$

Wave: $\quad \dfrac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2\, \Delta_h u_j^n$

(4)

Advection $\frac{u_{j+1}^n - u_j^n}{\Delta t} + \frac{s}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = 0$

Mode analysis

Fourier mode $u_j^n = a_n e^{ikx_j}$

k = wave number.

$$\frac{a_{n+1} - a_n}{\Delta t} + \frac{s}{2\Delta x} a_n \left(e^{ik\Delta x} - e^{-ik\Delta x}\right) = 0$$

$$a_{n+1} = a_n \cdot \left(1 - i\frac{\Delta t\, s}{\Delta x}\sin(k\Delta x)\right)$$

$$\lambda = \frac{\Delta t}{\Delta x} \cdot s = CFL$$

= dimensionless measure of wave speed.

amplification factor = symbol

$$= m(k) = \left(1 - i\lambda\sin(k\Delta x)\right)$$

stability / instability decided by

$$|m(k)| = \sqrt{m^* m}$$

$$= \sqrt{1 + \lambda^2 \sin^2(k\Delta x)}$$

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= larger than 1 for some k

= unstable.

Heat eqn 1-D

$$u_j^{n+1} = u_j^n + \frac{D\Delta t}{\Delta x^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

$$a_{n+1} = a_n + \frac{D\Delta t}{\Delta x^2}\left(2\cos(k\Delta x) - 2\right)$$

$\lambda = \frac{D\Delta t}{\Delta x^2}$ = dimensionless

m = real

$\max_k m(k)$ when $\cos = -1$,

$$|m| = |(1 - 4\lambda)| \leq$$

$$|m_{max}| > 1 \text{ if } \lambda > \tfrac{1}{2}$$

CFL condition: $\frac{D\Delta t}{\Delta x^2} \leq \frac{1}{2}$

$$\Delta t \leq \frac{\Delta x^2}{2D}$$

Wave eqn $\quad a_{n+1} = \left[2 + \frac{c^2\Delta t^2}{\Delta x^2}\left(2 - 2\cos(k\Delta x)\right)\right]a_n - a_{n-1}$

for each k there are 2 roots.

(b)

Stability: $\|u^n\|_{\ell^2} \le C(t_n)\|u^0\|_{\ell^2}$

or $\|u^n\|_{\ell^2} \le C(t_n)\left(\|u^0\| + \|u_1\|\right)$

Consistency + stability $\Rightarrow$ convergence (as before)