Section 5, Complex variable calculus

March 11, 2017, version 2.0

1 Introduction.

We now move from calculus of real functions of a real variable to complex analytic functions of a complex variable. This is *complex analysis*. Complex analysis will simplify some of our technical proofs. For example, it makes it easy to see that

$$\zeta(s) = \frac{1}{s-1} + f(s) \;, \quad \text{as } s \downarrow 1 \;,$$

where $f(s)$ is a differentiable function of s. The main point is as it was before: the Dirichlet series $\sum n^{-s}$ converges for $s > 1$ and goes to infinite as $s \downarrow 1$. But it will be easier to see that $f(s)$ is differentiable. This will make it easier to justify the crucial calculation

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum \Lambda(n) n^{-s} = \frac{1}{s-1} + g(s) \;,$$

where $g(s)$ is also differentiable near $s = 1$.

Complex contour integration and the Cauchy theorems for it make complex analysis powerful, not only as a technical tool. The prime number theorem is the statement

$$\sum_{n \leq x} \Lambda(n) = \psi(x) = x + o(X) \;, \quad \text{as } \; x \to \infty \;. \tag{1}$$

The function $\psi(x)$ can be represented as a contour integral in the complex plane. The "real" form of this integral is

$$\psi(x) = \frac{-1}{2\pi} x^{\sigma} \int_{-\infty}^{+\infty} \frac{e^{it\log(x)}}{\sigma + it} \frac{\zeta'(\sigma + it)}{\zeta(\sigma + it)} \, dt \;, \tag{2}$$

for $\sigma > 1$. The complex form, which soon should seem simpler, is

$$\psi(x) = \frac{-1}{2\pi i} \int_{-\sigma - i\infty}^{\sigma + i\infty} \frac{x^s}{s} \frac{\zeta'(s)}{\zeta(s)} \, ds \;. \tag{3}$$

It may be that Euler discovered the zeta function and the Euler product that connects $\zeta(s)$ to prime numbers. But Riemann discovered the integral formula (3) and recognized that it leads to a proof of the prime number theorem (1). Riemann's proof was incomplete,[1] mostly because the tools of modern complex analysis were unknown to him. Riemann's ideas were one of the big motivations for mathematicians to figure out complex analysis.

[1] See the beautiful discussion of Riemann's work in the book *Riemann's Zeta Function*, by Harold Edwards, formerly math professor at Courant.

2 The complex derivative.

A real function of a real variable is $f(x)$. A complex function of a complex variable is written $f(z)$. We write $z = x + iy$, with x and y being the real and imaginary parts of z. We write $f(z) = u(z) + iv(z) = u(x, y) + iv(x, y)$, depending on whether we want to focus on f as a function of the single complex variable z or on f as a function of the two real variables x and y.[2]

The complex plane is $\mathbb{C}$. A typical point $z \in \mathbb{C}$ is $z = x + iy$. Of course, $\mathbb{C}$ is the same as the real plane $\mathbb{R}^2$ with coordinates (x, y).[3] In complex calculus, it is common to replace $y = f(x)$ with $w = f(z)$. This is $w = u + iv = f(z) = f(x + iy)$. For example, if $f(z) = z^3$, then

$$\begin{aligned}
w &= z^3 \\
&= (x + iy)^3 \\
&= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\
&= x^3 + 3ix^2y + 3i^2xy^2 + i^3y^3 \\
&= x^3 + 3ix^2y - 3xy^2 - iy^3 \\
&= \left(x^3 - 3xy^2\right) + i\left(3x^2y - y^3\right) \ .
\end{aligned}$$

If we write $f(z) = z^3$ in the form $u + iv$, then $u(x, y) = x^3 - 3xy^2$ and $v = 3x^2y = y^3$.

The *modulus* of z is the length of the vector (x, y), which is the distance of the point (x, y) from the origin:

$$r = |z| = \sqrt{x^2 + y^2} \ .$$

The *argument* of z is that angle that the line from the origin to (x, y) makes with the x axis. This is written $\theta = \arg(z)$. The argument is not uniquely defined. If we "wind" a point z once around the origin, then the argument changes by 2π. It is common to assume that θ is chosen with $-\pi < \theta \leq \pi$, which puts a "branch cut" (the cut where the function $\theta(x, y)$ is discontinuous, something like the international date line) along the negative real axis. What is important is $r \geq 0$ and $(x, y) = (r\cos(\theta), r\sin(\theta))$.

We work a lot with the exponential e^a, where $a \in \mathbb{C}$. Here are the rules. This list is redundant, but (hopefuly) consistent.

- If a and b are any complex numbers, then

$$e^{a+b} = e^a = e^b \ .$$

- If y is a real number, then

$$e^{iy} = \cos(y) + i\sin(y) \ ,$$

[2] Traditional notation depends on the context. In analytic number theory, the zeta function is written $\zeta(s)$, where $s = \sigma + it$ There isn't a standard notation for $\mathrm{Re}(\zeta)$ and $\mathrm{Im}(\zeta)$.

[3] Some mathematicians disagree with this, calling $\mathbb{C}$ the *complex line*. They view $\mathbb{C}$ as one dimensional, even if that one dimension is complex.

where cos and sin are the real functions of a real argument y.

- If $z = x + iy$, then (the second line follows from the first)

$$\begin{aligned} e^z &= e^{x+iy} = e^x e^{iy} = e^x \cos(y) + ie^x \sin(y) \\ |e^z| &= e^x \quad , \quad \arg(e^z) = y + 2\pi k \ , \ \text{for some integer } k \ . \end{aligned}$$

- For any $z \in \mathbb{C}$ and $w \in \mathbb{C}$,

$$\begin{aligned} \sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \quad , \quad \cos(z) = \frac{e^{iz} + e^{-iz}}{2i} \\ e^{iz} &= \cos(z) + i \sin(z) \\ \sin(z + w) &= \sin(z)\cos(w) + \cos(z)\sin(w) \\ e^z &= 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \cdots \\ \sin(z) &= z - \frac{1}{6}z^3 + \cdots \\ \cos(z) &= 1 = \frac{1}{2}z^2 + \frac{1}{24}z^4 - \cdots \ . \end{aligned}$$

  You can derive the second and third formulas from the first. The first may be derived from the bottom three.

- If $z = x + iy$ with $r = |z|$ and $\theta = \arg(z)$, then

$$z = re^{i\theta} \ .$$

  Taking a different definition of the argument has the possible effect of replacing θ by $\theta \pm 2\pi$. This is OK here, because

$$e^{i(\theta+2\pi)} = e^{i\theta}e^{2\pi i} = e^{i\theta}\left(\cos(2\pi) + i\sin(2\pi)\right) = e^{i\theta}\left(1 + i0\right) = e^{i\theta} \ .$$

- If $a > 0$ and $z = x + iy$, then

$$a^z = e^{z\log(a)} = e^{x\log(a)}e^{iy\log(a)} = a^x e^{iy\log(a)} \ .$$

  These facts explain why the formulas (3) and (2) are the same.

The definition of derivative looks the same for complex and real functions:

$$\frac{df}{dz} = \frac{d}{dz}f(z) = f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \ . \tag{4}$$

To emphasize that z and h can be complex numbers, here is the definition written out with the ϵ δ definition of limit: For any real $\epsilon > 0$ there is a real $\delta > 0$ so that for any $h \in \mathbb{C}$ with $|h| \leq \delta$, we have (several properties of complex numbers are used to go from the first to the second)

$$\begin{aligned} \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| &\leq \epsilon \\ |f(z+h) - f(z) - hf'(z)| &\leq \epsilon \, |h| \ . \end{aligned}$$

In particular, if
$$f(z+h) - f(z) = ah + O(|h|^2) ,$$
then $f'(z) = a$. If there is such an a, or if the limit exists, then we say f is differentiable. A differentiable function of a complex variable is called *analytic*.[4] A complex function given by a formula is likely to be analytic. For example,
$$\frac{d}{dz} z^3 = 3z^2 .$$
There are more examples in the exercises.

Complex differentiation is not as much like ordinary differentiation as it may seem. We call complex differentiable functions "analytic" to emphasize the difference. If a real function is differentiable, we think of it as not being like $f(x) = |x|$ (with a corner at $x = 0$). We will see analyticity means much more. The *Cauchy Riemann* equations give a suggestion of the difference. You might think $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is differentiable if these partial derivatives all exist
$$\frac{\partial u(x,y)}{\partial x} \;, \quad \frac{\partial u(x,y)}{\partial y} \;, \quad \frac{\partial v(x,y)}{\partial x} \;, \quad \frac{\partial v(x,y)}{\partial y} \;.$$
But that would be wrong. There are different expressions for $f'(z)$ in terms of partial derivatives of u and v depending on the direction with which $h \to 0$ in (4). These expressions are equal if the complex limit (4) exists. If $h \to 0$ on the real axis we write $h = \Delta x \in \mathbb{R}$ and calculate
$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x,y) - iv(x,y)}{\Delta x} \\ &= \frac{\partial u(x,y)}{\partial x} + i\frac{\partial v(x,y)}{\partial x} \;. \end{aligned}$$
If $h \to 0$ on the imaginary axis, we write $h = i\Delta y$ and calculate
$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{u(x, y+i\Delta y) + iv(x, y+i\Delta y) - u(x,y) - iv(x,y)}{i\Delta y} \\ &= i\frac{\partial u(x,y)}{\partial y} - \frac{\partial v(x,y)}{\partial y} \;. \end{aligned}$$
If these two expressions for f' are equal, we equate the real and imaginary parts to get the *Cauchy Riemann* equations
$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= -\frac{\partial v(x,y)}{\partial y} \\ \frac{\partial v(x,y)}{\partial x} &= \frac{\partial u(x,y)}{\partial y} \end{aligned} \tag{5}$$
A good book on complex analysis has much more interesting discussion of complex functions and their derivatives. The geometry of complex functions as mappings $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is interesting. But this is a course on number theory, so we do this analysis as it comes up in the problems we're working on.

[4]This originally meant that it is possible to analyze f using complex analysis.

3 Contour integrals

The ordinary integral undoes the ordinary derivative. The complex contour integral undoes the complex derivative. Suppose $f(x)$ is a real function of a real variable. You can integrate the derivative or differentiate the integral and get back the original function. The definite integral of the derivative:

$$\int_a^b f'(x)\, dx = f(b) - f(a) \; .$$

Differentiating the indefinite integral, if

$$F(x) = \int_a^x f(y)\, dy \; ,$$

Then

$$F'(x) = f(x) \; .$$

These are forms of the fundamental theorem of calculus.

The complex contour is an integral along a path, or *contour*, in the complex plane. A contour, which is a one dimensional continuous curve in $\mathbb{C}$, may be defined as the image of a real interval $[a,b] \subset \mathbb{R}$, where the point $t \in [a,b]$ is mapped to the point $\zeta(t) \in \mathbb{C}$ defined by[5] $\zeta(t) = \xi(t) + i\eta(t)$. For example, here is a contour that goes once around the unit circle: $\zeta(t) = e^{it} = \cos(t) + i\sin(t)$, for $0 \leq t \leq 2\pi$. Here is a different contour that goes around the same circle the other way: $\zeta(t) = e^{-it}$ for $0 \leq t \leq 2\pi$. These contours start and end at the same place, but they take different routes and therefore are different contours. A contour integral over the counter-clockwise contour (e^{it}) may be different from the integral over the clockwise contour (e^{-it}). It is common to define a contour in pieces. For example here is a contour that starts at $\zeta(0) = 0$ and ends at $\zeta(2) = 1 + i$ but gets there by first moving horizontally then vertically

$$\zeta(t) = \begin{cases} t & \text{if } 0 \leq t \leq 1 \\ 1 + i(t-1) & \text{if } 1 \leq t \leq 2 \; . \end{cases}$$

You can combine two contours ζ_1 defined on $[a_1, b_1]$ and ζ_2 defined on $[a_2, b_2]$ into a single one if ζ_2 starts where ζ_1 ends (in formulas: $\zeta_1(b_1) = \zeta_2(a_2)$). The idea is simple – first do ζ_1 then do ζ_2, but the formula is clunky:

$$\zeta(t) = \begin{cases} \zeta_1(t) & \text{if } a_1 \leq t \leq a_2 \\ \zeta_2(t - a_2 + b_1) & \text{if } b_1 \leq t \leq b_1 + (b_2 - a_2) \; . \end{cases}$$

We use a letter like Γ to denote the contour, and ζ to denote the parametrization of it. The "parameter" in "parametrization" is t. We write a point z on the contour as a function of the parameter t. It is important that Γ is not just

[5] It is common to use Greek letters that seem to correspond to Latin letters. Here, ζ is a z value, ξ is an x value and η is a y value. I think the association $\xi \leftrightarrow x$ is incorrect. It should be χ. But mathematicians use ξ more often. Sorry, Greeks.

a subset of $\mathbb{C}$, but is a path. If ζ_1 and ζ_2 cover the same points in $\mathbb{C}$ in the same order, then we say that they represent the same contour. An example of this is the two contours $\zeta_1(t) = e^{it}$ for $0 \leq t \leq 2\pi$ and $\zeta_t(t) = e^{it^2}$ for $0 \leq t \leq \sqrt{2\pi}$. On the other hand, the contour $\zeta_3(t) = e^{it}$ for $0 \leq t \leq 4\pi$ is different because it "winds around" the origin twice, rather than once. The definition of *reparametrization* is that there is a continuous monotone function $s = r(t)$, with $r(a_1) = a_2$ and $r(b_1) = b_2$), so that $\zeta_2(r(t)) = \zeta_1(t)$. For the example contours just given, $a_1 = a_2 = 0$, $b_1 = 2\pi$, and $b_2 = \sqrt{2\pi}$, and $r(t) = \sqrt{t}$.

The integral of a continuous real function over the real interval $[a, b]$ is defined using a sequence of *partitions*. A partition is a sequence $a = x_0 < x_1 < \cdots < x_n = b$. The maximum spacing of a partition is

$$M = \max_{0 \leq k < n} \Delta x_k = \max_{0 \leq k < n} x_{k+1} - x_k \ .$$

The definition is[6]

$$\int_a^b f(x)\,dx = \lim_{M \to 0} \sum_0^{n-1} f(x_k)\Delta x_k \ .$$

In this limit, n and M depend on the partition. We take a sequence of partitions so that $M \to 0$ and $n \to \infty$. The limit exists and the result is the same for any sequence of partitions as long as $M \to 0$.

The complex contour integral can be defined in a similar way. A partition of a contour is a sequence of points on the contour in the order of the contour: $z_k = \zeta(t_k)$ with $z_0 = \zeta(a)$, $z_n = \zeta(b)$, and $t_{k+1} > t_k$. The maximum spacing is

$$M = \max_{0 \leq k < n} |\Delta z_k| = \max_{0 \leq k < n} |z_{k+1} - z_k| \ . \tag{6}$$

The contour integral is

$$\int_\Gamma f(z)\,dz = \lim_{M \to 0} f(z_k)\Delta z_k \ . \tag{7}$$

If ζ_2 is a reparametrization of ζ_1, so that they define the same Γ, then the definitions (6) and (7) are the same. The parameter values t_k and s_k, with $\zeta_1(t_k) = z_k$ and $\zeta_2(s_k) = z_k$, are different but related by the reparametrizing function r. If $f(z)$ is a continuous function of z for z in the contour, then the limit as $M \to 0$ exists. The proof from mathematical analysis that worked for real integrals of a real variable works here.

We are ready for a form of the fundamental theorem of calculus for contour integrals: If Γ is a contour from $u = \zeta(a)$ to $v = \zeta(b)$, and if the hypotheses described next are hold, then

$$\int_\Gamma f'(z)\,dz = f(v) - f(u) \ . \tag{8}$$

[6] The sum on the right is a *Riemann sum*. Riemann's contribution to mathematics include the zeta function tricks, the definition of the Riemann integral, and much more.

The main hypothesis is that $f(z)$ is uniformly differentiable with a continuous derivative. There is a function $f'(z)$, defined for all z in the contour. Since the points of the contour form a compact set in $\mathbb{C}$ because they are the image of the compact set $[a, b]$ under the continuous function ζ. This means that a continuous function on the contour, such as $f'(z)$ is bounded. Uniformly differentiable, like uniformly continuous, means that the same δ works for every z in the contour:

$$\begin{aligned}&\text{For any } \epsilon > 0 \text{ there is } \delta > 0 \text{ so that}\\ &|f(z+h) - f(z) - hf'(z)| \leq \epsilon\,|h| \\ &\text{ if } \ |h| \leq \delta \ ,\end{aligned} \tag{9}$$

The other hypothesis is that the contour is *rectifiable*, which means that there is an L so that for any partition,

$$\sum_{k=0}^{n-1} |z_{k_1} - z_k| \leq L \ . \tag{10}$$

The smallest L (the inf of all L) may be thought of as the length of Γ. Γ is rectifiable if it has a parameterization ζ that is a continuously differentiable function of t. If you combine two rectifiable contours you get a rectifiable contour. All the contours we use are combinations of continuously differentiable ones, so they are all rectifiable. You can find examples of non-rectifiable contours in the Wikipedia page on fractals.

Here is the proof of the fundamental theorem (8) using the hypotheses (9) and (10). Choose $\epsilon > 0$ and take δ so that inequality (9) is satisfied. Choose a partition on the contour so that $|z_{k+1} - z_k| \leq \delta$ for all k. Suppose the points z_k form a partition with $M \leq \delta$. The inequality (9) may be written as

$$|f(z_{k+1}) - f(z_k) - f'(z_k)\Delta z_k| \leq \epsilon\,|z_{k+1} - z_k| \ .$$

We will add these inequalities and use cancellations of the form

$$[f(z_{k+2}) - f(z_{k+1})] + [f(z_{k+1}) - f(z_k)] = [f(z_{k+2}) - f(z_k)] \ .$$

We use these inequalities for $k = 0, 1, \ldots, n-1$, and the fact that $z_0 = u$ and $z_n = v$ and we get

$$\left| f(v) - f(u) - \sum_{k=0}^{n-1} f'(z_k)\Delta z_k \right| \leq \epsilon \sum_{k=0}^{n-1} |\Delta z_k| \leq \epsilon L \ .$$

The second inequality uses the inequality (10) that expresses the hypothesis that the contour is rectifiable. As we let $M \to 0$ and $\epsilon \to 0$ and $\delta \to 0$ (these are not independent), the right side converges to zero and the sum on the left side converges to the contour integral. Therefore

$$\left| f(v) - f(u) - \int_\Gamma f'(z)\, dz \right| = 0 \ ,$$

which is the fundamental theorem (8).

Sometimes we calculate a contour integral using the parameterization ζ. If ζ' exists, and if the rules of integral calculus apply, we should be able to write $dz = \zeta'(t)dt$, and then

$$\int_\Gamma f(z)\,dz = \int_a^b f(\zeta(t))\zeta'(t)\,dt\ . \tag{11}$$

The integral on the right is an ordinary (Riemann) integral of the function $g(t) = f(\zeta(t))\zeta'(t)$ with respect to the real variable t. It doesn't matter that $g(t)$ has complex values. It is possible to prove (11) directly from the limit definitions of the integral. Suppose $a = t_0 < t_1 < \cdots < t_n = b$ is a partition of the interval $[a, b]$ and $z_k = \zeta(t_k)$ is the corresponding partition of the contour. Then if $\Delta t_k \leq \delta$ then

$$|\Delta z_k - \zeta'(t_k)\Delta t_k| \leq \epsilon \Delta t_k\ .$$

The difference between the Riemann sum approximations to the integrals is

$$\left|\sum_{k=0}^{n-1} f(z_k)\big(\Delta z_k - \zeta'(t_k)\Delta t_k\big)\right| \leq C\epsilon \sum_{k=0}^{n-1} \Delta t_k\ .$$

The rest is an exercise, literally.

Parametrized contour integration is a way to get one of the most important formulas of contour integration and all of complex analysis. We integrate on the closed contour parametrized by $\zeta(t) = re^{it}$, (with $d\zeta = ire^{it}dt$) for $0 \leq t < 2\pi$. This is the circle of radius r around the origin, traced in the counter-clockwise direction. It is often written

$$\int_0^{2\pi} f(re^{it})ire^{it}dt = \oint_{|z|=r} f(z)\,dz\ .$$

The circle through $\int$ (giving $\oint$) tells us that the contour is closed. The radius r circle, centered about 0 and traced once around in the counter-clockwise direction, is $|z| = r$. The very special important integral is

$$\oint_{|z|=r} z^{-1}\,dz = \int_0^{2\pi} r^{-1}e^{-it}ire^{it}dt = 2\pi i\ . \tag{12}$$

The integral is independent of r because of the Cauchy integration theorem, and also because the function is on the order of r^{-1} and the contour has length r, so the factors of r cancel. If we would integrate z^n on $|z| = r$, we also would know that the integral is independent of r by Cauchy's theorem. On the other hand, the integral would be proportional to r^{n+1} (r^n from z^n and r from the length of the contour). The only way this is possible is for the integral to be equal to zero.

We end this section with a technicality, a version of the intermediate value theorem for complex differentiable functions. Suppose $\Omega \subset \mathbb{C}$ is an open set and that $f(z)$ is defined and differentiable for every $z \in \Omega$. Suppose $u \in \Omega$ and $v \in \Omega$ and the the straight line contour parameterized by $\zeta(t) = u + t(v - u)$ on $[0, 1]$ is contained in Ω. Then

$$|f(v) - f(u)| \leq \left(\max_{\Gamma} |f'(z)| \right) |v - u| \ . \tag{13}$$

You can prove this using the fundamental theorem and the parametrized contour integral (11)

$$|f(v) - f(u)| = \left| \int_{\Gamma} f'(\zeta(t)) \zeta'(t) dt \right| \ .$$

For our contour, $\zeta'(t) = v - u$, so

$$|f(v) - f(u)| \leq \left(\max_{\Gamma} |f'(z)| \right) \int_0^1 |v - u| \ dt \ .$$

This is the inequality (13). The technicalities, the domain Ω that must contain the line segment Γ, will be important later. For example, there is something called a *branch cut*, which could be the negative real axis in the complex plane. That is, $z \in \Omega$ unless z is real and $z \leq 0$. The function $f(z) = \sqrt{(z)}$ can be defined on Ω but not on all of $\mathbb{C}$. The value of $\sqrt{(z)}$ is different if you approach the branch cut from above or below. If $\sqrt{z} > 0$ for $z > 0$ real, then, taking $\epsilon > 0$ to be real,

$$\lim_{\epsilon \downarrow 0} \sqrt{-1 + i\epsilon} = i$$
$$\lim_{\epsilon \downarrow 0} \sqrt{-1 - i\epsilon} = -i$$

If $u = -1 - \epsilon$ and $v = -1 + \epsilon$ then $|v - u| \to 0$ as $\epsilon \to 0$ but $|f(v) - f(u)| \to 2 \neq 0$. The shortest contour from u to v that stays in Ω is not short as $\epsilon \to 0$.

4 Deforming contours

In the time of Euler, a mathematician might have calculated the integral

$$\frac{2}{3} = \int_{-1}^{1} x^2 \, dx \ .$$

in the following way. Use a substitution $x = e^{it}$, so that $x = -1$ when $t = -\pi$ and $x = 1$ when $t = 0$. The differential is $dx = ie^{it}dt$. Therefore

$$\int_{-1}^{1} x^2 \, dx = \int_{-\pi}^{0} \left(e^{it} \right)^2 ie^{it} \, dt \ .$$

It is an exercise to calculate the right side and show that the answer is correct. According to our definitions of contour integrals, this is the integral of $f(z) = z^2$ over two distinct contours:

$$\begin{aligned} \Gamma_r = & \text{ real interval } [-1, 1] \ , && \text{parameterized by } \zeta_r(t) = t \text{ on } [-1, 1] \\ \Gamma_+ = & \text{ upper half circle } , && \text{parameterized by } \zeta_+(t) = e^{it} \text{ on } [-\pi, 0] \ . \end{aligned}$$

For us, it may not be obvious that

$$\int_{\Gamma_r} z^2\, dz = \int_{\Gamma_+} z^2\, dz \; .$$

In fact, different contours from u to v can give different contour integrals. Let Γ_+ be as above, and let Γ_- be a different route from -1 to 1:

$$\Gamma_- = \text{ lower half circle }, \quad \text{parameterized by } \zeta_-(t) = e^{-it} \text{ on } [-\pi, 0] \; .$$

Then

$$\begin{aligned}
\int_{\Gamma_+} z^{-1}\, dz &= \int_{-\pi}^{0} e^{-it} i e^{it}\, dt \\
&= \int_{-\pi}^{0} idt \\
&= \pi i \; .
\end{aligned}$$

Integrating on the other contour gives

$$\begin{aligned}
\int_{\Gamma_-} z^{-1}\, dz &= \int_{-\pi}^{0} e^{it} - ie^{-it}\, dt \\
&= \int_{-\pi}^{0} -idt \\
&= -\pi i \; .
\end{aligned}$$

The conclusion is clear. Sometimes the integrals over two paths from u to v are equal. Sometimes they aren't.

We now know (thanks to Euler) that integrating on different contours gives the same answer if you can *deform* one contour into the other one. Suppose Γ_0 and Γ_1 are two contours from u to v. A *deformation* of Γ_0 to Γ_1 is a continuous family of contours Γ_s defined for $0 \leq s \leq 1$ so that you get Γ_0 for $s = 0$ and Γ_1 for $s = 1$. In topology, a deformation like this is called a *homotopy*.

The deformations we use will be simple and explicit. The idea will be clear if we consider a family of parameterizations $\zeta(t, s)$, defined for $t \in [a, b]$ and $s \in [0, 1]$. For each s, the function of t given by $\zeta(t, s)$ parameterizes the contour Γ_s. We assume that $\zeta(t, s)$ is twice continuously differentiable as a function of the two variables t and s. At some point in the calculation below, we use the equality of mixed partials:

$$\frac{\partial}{\partial s}\left(\frac{\partial \zeta(s,t)}{\partial t}\right) = \frac{\partial^2 \zeta}{\partial s \partial t} = \frac{\partial^2 \zeta}{\partial t \partial s} = \frac{\partial}{\partial s}\left(\frac{\partial \zeta(s,t)}{\partial t}\right) \; .$$

If we have a deformation like this, we can do the calculation (below) to see that

$$\frac{d}{ds} \int_{\Gamma_s} f(z)\, dz = 0 \; . \tag{14}$$

In particular, if Γ_0 can be connected to Γ_1 with such a differentiable deformation, then

$$\int_{\Gamma_0} f(z)\,dz = \int_{\Gamma_1} f(z)\,dz\;. \tag{15}$$

This is the *Cauchy integral theorem*, which Euler knew before Cauchy.

The important thing usually isn't the contours, but the function f. It is necessary that $f(\zeta(s,t)$ should be differentiable for every s and t in this family of deformed contours. For example, we can deform Γ_- to Γ_+ using the deformation

$$\zeta(s,t) = s\zeta_+(t) + (1-s)\zeta_-(t)\;.$$

This "straight line homotopy" (it's a straight line as a function of s, not as a function of t) works for any two parameterized contours. The problem is that there might be an s so that Γ_s runs through a z value where f is not defined. Take the straight line homotopy from Γ_- to Γ_+ and the function $f(z) = z^{-1}$. For $s = \frac{1}{2}$, this gives $\zeta(\frac{1}{2}, t) = \cos(t)$ (check this) on the interval $-\pi \leq t \leq 0$. For $t = -\frac{\pi}{2}$, this gives $\zeta(\frac{1}{2}, \frac{-\pi}{2}) = \cos(-\frac{\pi}{2}) = 0$, where z^{-1} is not defined. This cannot be fixed with a better proof, as we know the Γ_+ and Γ_- integrals are different.

The proof of (15) is a calculation (explanations follow the calculation):

$$\begin{aligned}
\frac{d}{ds}\int_{\Gamma_s} f(z)\,dz &= \frac{d}{ds}\int_a^b f(\zeta(t,s))\frac{\partial\zeta(t,s)}{\partial t}\,dt \\
&= \int_a^b \frac{\partial}{\partial s}\left(f(\zeta(t,s))\frac{\partial\zeta(t,s)}{\partial t}\right)dt \\
&= \int_a^b \left(f'(\zeta(t,s))\frac{\partial\zeta(t,s)}{\partial s}\frac{\partial\zeta(t,s)}{\partial t} + f(\zeta(t,s)\frac{\partial^2\zeta(t,s)}{\partial s\partial t}\right)dt \\
&= \int_a^b \frac{\partial}{\partial t}\left(f(\zeta(t,s)\frac{\partial\zeta(t,s)}{\partial s}\right)dt \\
&= f(v)\frac{\partial\zeta(1,s)}{\partial s} - f(u)\frac{\partial\zeta(0,s)}{\partial s} \\
&= 0\;.
\end{aligned}$$

To go from line 1 to line 2, we moved the derivative with respect to s inside the integral. It is written as a total derivative on line 1 and a partial derivative with respect to s on line 2, because the integrand is a function of two variables. To go from line 2 to line 3 we differentiate using the product rule and then the chain rule. The chain rule for $f(\zeta(s,t))$ is

$$\frac{\partial}{\partial s}f(\zeta(s,t)) = f'(\zeta(s,t))\frac{\partial\zeta}{\partial s}\;. \tag{16}$$

This is an exercise. Line 4 equals line 3 for the same reason line 2 equals line 3, because the s and t partial derivatives commute. Line 4 to line 5 is the ordinary fundamental theorem of calculus, plus the fact that ζ goes from u to

v, so $\zeta(0,s) = u$ for all s, etc. The last step is the fact if $\zeta(0,s) = u$ for all s, then $\frac{\partial\zeta}{\partial s} = 0$.

There is another statement of the Cauchy integral theorem that refers to closed contours that are *simple* or *contractable*. A contour is *closed* if it starts and ends at the same place. In terms of a parameterization, this means $\zeta(a) = \zeta(b)$. A contour is *simple* if it has a parameterization with $\zeta(t_1) \neq \zeta(t_2)$ for $a \leq t_1 < t_2 < b$. This means it never touches itself. A simple closed contour divides $\mathbb{C}$ into an "interior" and an "exterior", which is not so hard to prove if the contour is rectifiable but pretty hard to prove if it is just continuous (i.e., has a continuous parameterization). The interior plus the contour itself form a compact set. Suppose that $f(z)$ is differentiable in a neighborhood of this compact set. The Cauchy theorem states that

$$\oint_\Gamma f(z)\,dz = 0\ .$$

The circle in the integral sign indicates that it's a closed contour.

You can prove this form of the Cauchy theorem by invoking Green's theorem (which is the two dimensional version of Stokes' theorem and the two dimensional version of the divergence theorem, in case you've never heard of it). This theorem, also is either a little informal or rests on a significant foundation of topology. Suppose the parameterization is $\zeta(t) = \xi(t) + i\eta(t)$ and the differentiable function is $f(z) = u(x,y) + iv(x,y)$. The contour integral, written in a combination of old and less old notation, is

$$\begin{aligned}
\oint_\Gamma f(z)\,dz &= \oint_\Gamma (u(\xi(t),\eta(t)) + iv(\xi(t),\eta(t))(\xi'(t) + i\eta'(t))\,dt \\
&= \oint_\Gamma (u+iv)(dx+idy) \\
&= \oint_\Gamma (udx - vdy) + i\oint_\Gamma (vdx + udy)\ .
\end{aligned}$$

Green's theorem says the first integral is zero because

$$\mathrm{curl}(u,v) = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0\ ,$$

which is one of the Cauchy Riemann equations. The second integral is zero because of the other Cauchy Riemann equation

$$\mathrm{div}(u,v) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\ .$$

As an example, let the contour be the circle of radius r about the origin that starts at $a = r$ (on the positive real axis) and ends at the same point $b = r$. A contour that starts and ends at the same place is a *closed* contour. We will see that it is very common to be able to evaluate integrals around closed contours. If $f(z)$ is analytic everywhere inside the contour, we may deform the contour to a single point without changing the integral. Therefore, the integral around such a contour is zero. But the integral may not be zero if $f(z)$ is *singular* (not analytic) somewhere inside the contour. For example, the function $f(z) = z^{-1}$ is analytic except at $z = 0$, so it is not analytic at every point inside the circle of radius r. Integration around a closed contour is often written $\oint$. We calculate

$$\oint_{|z|=r} z^{-1}\, dz \ . \tag{17}$$

This notation is ambiguous because there are two ways to go around the contour, clockwise and counter-clockwise. One generally should say which is intended, but one often forgets to say if the direction is counter-clockwise. We use the parametrization $z = re^{i\theta}$, which has $dz = rie^{i\theta}d\theta$, and $z^{-1} = r^{-1}e^{-i\theta}$. The integral (17) is

$$\int_0^{2\pi} r^{-1}e^{-i\theta}\, rie^{i\theta}\, d\theta = 2\pi i \ .$$

The factors of r cancel, making the integral independent of the contour, as it should be.

5 Some functions

Most of the functions $f(z)$ we use are either these examples or are built from them as infinite sums or products.

$\mathbf{e^{ikz}}$. On the real axis, this is purely oscillatory (it just goes round and round), because $e^{ikx} = \cos(x) + i\sin(x)$. Off the real axis it grows or decays "exponentially". If $z = x + iy$, then

$$e^{ikz} = e^{ikx}e^{-ky} \ .$$

If $k > 0$, then this decays for positive y and grown for negative y. If $k < 0$ it does the opposite. The behavior of e^{ikz} as you leave the real axis depends on the sign of k. The functions $f(z) = \cos(z)$ and $f(z) = \sin(z)$ grow exponentially off the real axis no matter which way you go because they have complex exponentials of both signs.

$\mathbf{z^p}$. You can write z in "polar coordinates" as $z = re^{i\theta}$. This is the same as $x = r\cos(\theta)$ and $y = r\sin(\theta)$. It is understood that $r = |z| \geq 0$. However, the *argument*, θ is not completely determined. If $z = re^{i\theta}$, then $z = re^{i(\theta+2\pi)}$. It is a common convention to take $-\pi < \theta \leq \pi$. If you do this, then θ is a discontinuous function of z, discontinuous at every point of the "branch cut" on the negative real axis. For any real p, you can try to define

$$z^p = r^p e^{ip\theta} \ .$$

If p is an integer (positive or negative), this gives a unique answer, regardless of which θ you pick to represent z. If p is not an integer, the answer depends on which θ you pick. There is no "single valued" function (the definition of function includes being single valued, only one value of $f(z)$ for each z, but complex analysis people say this anyway) z^p. Take $f(z) = \sqrt{z}$ as an example. We can define it for all z using $\sqrt{z} = \sqrt{r}e^{\frac{i\theta}{2}}$, with $-\pi < \theta \leq \pi$, but this function is not continuous, and certainly not differentiable, for $z = -1$, or any other point on the negative real axis.

$\mathbf{n^s}$. Here, we write the complex variable as $s = \sigma + it$. In exponential notation, this is

$$n^s = n^{\sigma+it} = n^\sigma n^{it} = n^\sigma e^{it\log(n)} \ .$$

This satisfies $|n^s| = n^\sigma$. Therefore, the Dirichlet series

$$\zeta(s) = \sum_1^\infty n^{-s} \tag{18}$$

converges absolutely if $\sigma > 1$. But it's much more "interesting" to figure out what happens with σ is fixed and $t \to \infty$.

6 The Cauchy integral formula and consequences

There are big differences between the possible behaviors of real differentiable functions and complex differentiable functions. Many of these differences are consequences of the *Cauchy integral formula*

$$f(z) = \frac{1}{2\pi i}\oint_\Gamma \frac{f(w)}{w-z}\,dw \ . \tag{19}$$

In the integral, w is the integration variable that we called z in the previous sections and z is a parameter. The formula is valid (as we will soon see) if Γ is a closed contour, such as a circle, that "winds around" z exactly once in the counter-clockwise direction, and if f is differentiable inside Γ. A course on complex analysis would explore the topological concepts of "winding" and "inside" at length. But this course on analytic number theory will just suppose Γ is a circle in the complex plane with radius r_0 centered at a point z_0, and that $|z - z_0| < r$, and that f is analytic for all z in some bigger disk of the form $|z - z_0| < r_1$ with $r_1 > r$.

The proof of the Cauchy formula (19) uses tricks that are used constantly in complex analysis. The contour Γ may be parametrized by $\zeta(t) = z_0 + r_0e^{it}$ for $0 \leq t \leq 2\pi$. It may be deformed to a small contour about z, which we call Γ_δ parametrized by $\zeta_\delta(t) = z + \delta e^{it}$. We will take the limit $\delta \to 0$. We assume $\delta < r_0 - |z - z_0|$. You can draw a picture to see that $\delta < r_0 - |z - z_0|$ insures that all of Γ_δ is inside Γ. One deformation of Γ to Γ_δ is the family of straight line deformation

$$\zeta(t,s) = (1-s)\zeta(t) + s\zeta_\delta(t) = (1-s)\left[z_0 + r_0e^{it}\right] + s\left[z + \delta e^{it}\right] \ .$$

All the points of $\zeta(t,s)$ for all $t \in [0,2\pi]$ and $s \in [0,1]$ are in Γ, so f is differentiable (in the complex sense). Therefore (put in the parametrized integral on the second line)

$$\begin{aligned}
\frac{1}{2\pi i}\oint_\Gamma \frac{f(w)}{w-z}\,dw &= \frac{1}{2\pi i}\oint_{\Gamma_\delta} \frac{f(w)}{w-z}\,dw \\
&= \frac{1}{2\pi i}\int_0^{2\pi i} \frac{f(z+\delta e^{it})}{\delta e^{it}}\delta e^{it}\, idt \\
&= \frac{1}{2\pi}\int_0^{2\pi i} f(z+\delta e^{it})dt \; .
\end{aligned}$$

Now, since $f(w)$ is continuous near $w = z$, for any $\epsilon > 0$ there is a $\delta > 0$ so that

$$\max_{|z-w|\leq\delta} |f(w) - f(z)| \leq \epsilon \; .$$

Therefore,

$$\left| \frac{1}{2\pi}\int_0^{2\pi i} f(z+\delta e^{it})dt - f(z) \right| \leq \epsilon \; .$$

This is the proof of the Cauchy integral formula (19), let $\epsilon \to 0$.

We can use the Cauchy formula (19) to see that complex differentiable functions of a complex variable are nothing like real differentiable functions of two real variables (x,y). For example, the function $f(x,y) = 1 - x^2 - y^2$ is differentiable in the real derivative sense. But functions differentiable in the complex sense satisfy the *maximum principle*

$$|f(z)| \leq \max_{|w-z|=r} |f(w)| \; . \tag{20}$$

This is not true about $f(x,y) = 1 - x^2 - y^2$, since you can take $z = 0$ and $r = 1$ and get $1 \leq 0$. You prove this by taking Γ with $z = z_0$ and putting absolute values in the Cauchy formula (19).

As an example, take $f(w) = \cos(w) = \frac{1}{2}\left(e^{iw} + e^{-iw}\right)$. Then take $z = 0$ and $r = \frac{\pi}{2}$. If you stick to the real axis, you're trying to bound $\cos(0) = 1$ by $\cos(\pm\frac{\pi}{2}) = 0$. This doesn't work. But if you go off the real axis, you can calculate, for example

$$\cos(\tfrac{i\pi}{2}) = \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{\frac{-\pi}{2}}\right) = 2.51 > 1 \; .$$

In this case, the right side of (20) is larger than the left side. It's a theorem, which we won't prove, that the two sides are equal only if f is constant.

You can differentiate the Cauchy formula (19) with respect to z and get formulas for f', f'', etc. We assumed that f is a differentiable function of z, so it may not be surprising to have a formula for f'. But we did not assume that f'' exists. The Cauchy formula proves f'' exists (details below). Real differentiation is different. Let $f(x)$ be the function with $f(x) = 0$ for $x < 0$ and

$f(x) = x^2$ for $x \geq 0$. This function is differentiable with derivative $f'(x) = 0$ for $x < 0$ and $f'(x) = x$ for $x \geq 0$. But the second derivative at $x = 0$ does not exist, because (since $f'(0) = 0$)

$$\frac{d}{dx} f'(0) = \lim_{h \to 0} \frac{f'(h)}{h}$$

This limit does not exist because the right side is 1 if $h > 0$ and it is 0 if $h < 0$. You will be frustrated if you try to do this with a complex differentiable function. It's impossible.

You find the derivative formulas by differentiating the integral in (19) with respect to z. The first result is

$$f'(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(w)}{(w-z)^2} \, dw \; . \tag{21}$$

You can do this repeatedly to get

$$f''(z) = \frac{2}{2\pi i} \oint_\Gamma \frac{f(w)}{(w-z)^3} \, dw \tag{22}$$

$$\vdots$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(w)}{(w-z)^{n+1}} \, dw \; . \tag{23}$$

This implies that if a complex function is differentiable (in a disk of radius r about the point z), then it has all higher derivatives also.

The formulas (23) lead to a family inequalities like the maximum principle, but for derivatives. These allow us to show, for example, that if $f_n(z)$ is a family of complex differentiable functions that converges uniformly to $f(z)$, then f' converges to f' and similarly for higher derivatives. If

$$f(z) = \sum_1^\infty a_n(z)$$

and the sum converges uniformly and absolutely, then

$$f'(z) = \sum_1^\infty a_n'(z) \; ,$$

with the sum also converging uniformly and absolutely. This, also, is not true about real differentiable functions. Consider the sum

$$f(x) = \sum_1^\infty \frac{1}{n^2} \sin(n^2 x) \; . \tag{24}$$

The sum converges absolutely and uniformly, but the derivative sum does not converge

$$f'(x) \quad ? \stackrel{?}{=} ? \quad \sum_1^\infty \cos(n^2 x) \; .$$

The sum on the right is infinite if $x = 0$. If you look in the complex plane, the sum

$$\sum_1^\infty \frac{1}{n^2} \sin(n^2 z)$$

does not converge uniformly in any disk in the complex plane.

For the bounds on derivatives, unlike the maximum principle, it matters how close z is to the boundary of a disk of radius r about z_0. Suppose $f(z)$ is differentiable in a disk of radius r about a point z_0. This implies that $f(z)$ is a continuous function of z so this maximum is achieved:

$$M = \max_{|z-z_0|\leq r} |f(z)| \ . \tag{25}$$

Let Γ be the contour parameterized by $\zeta(t) = z_0 + re^{it}$. Then the integral representation (21) leads to

$$\begin{aligned}
f'(z) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{it})}{z_0+re^{it}-z} ie^{it}dt \\
|f'(z)| &\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|f(z_0+re^{it})\right|}{|z_0+re^{it}-z|}dt \\
&\leq M \max \frac{1}{|z_0+re^{it}-z|}\frac{1}{2\pi}\int_0^{2\pi} dt \qquad (26) \\
|f'(z)| &\leq \frac{M}{r-|z-z_0|} \ . \qquad (27)
\end{aligned}$$

The last step uses some plane geometry. The denominator in (26) is the distance between a point $w - z_0 + re^{it}$ on Γ and z inside Γ. A drawing should convince you that the closest distance is $r - |z - z_0|$. This is positive because $|z - z_0| < r$, which is what it means that z is inside Γ. You get the inequality (27) by noting that you maximize the fraction by minimizing the denominator.

As another example, consider the function $f(z) = \sqrt{z}$ defined for $|z - 1| \leq 1$. Let Γ be the contour centered at $z_0 = 1$ with radius $r = 1$. For this function and contour, $M = \sqrt{2}$ (where is this attained?) and $|f(z)| \leq \sqrt{2}$ for all z with $|z| \leq 1$. The derivative is $f'(z) = \frac{1}{2}z^{-\frac{1}{2}}$. This has $|f'(z)| \to \infty$ as $z \to 0$, which is a point of Γ. The derivative bound (27) allows this possibility.

These ideas apply to differentiation of series. Suppose the functions $a_n(s)$ are complex differentiable functions and are bounded in a disk of radius r about s_0 with bounds

$$M_n = \max_{|s-s_0|=r} |a_n(s)| \ .$$

(We use s instead of z for the complex variable because we're about to apply this to the Dirichlet series (18)) As in the dominated convergence theorem, suppose

$$\sum_1^\infty M_n < \infty \ .$$

Now let $r_1 < r$ be the radius of a smaller disk about s_0. Then the derivative bound (27) gives

$$M'_n = \max_{|s-s_0| \leq r_1} |a'_n(s)| \leq \frac{M_n}{r - r_1} \; .$$

This implies that the derivative sum is also absolutely convergent and subject to the dominated convergence theorem we proved before. We don't have to calculate $a'(s)$ to know this. For example, we know that if $s = \sigma + it$ and if $\sigma > 1$, then

$$\zeta'(s) = -\sum_1^\infty \log(n) n^{-s}$$

is absolutely convergent. We learn that $\zeta(s)$ is a complex differentiable function of s for $\sigma > 1$. The details are an exercise.

7 Taylor series

If $f(x)$ is a real function of a real variable x, we say f is *analytic* at x_0 if there is an $r_1 > 0$ so that if $|x - x_0| \leq r_1$, then

$$f(x) = \sum_0^\infty \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} \; ,$$

A complex function of a complex variable is called analytic if its Taylor series converges in this way. A complex function of a complex variable is called analytic at z_0 if there is an r so that $f'(z)$ exists for all $|z - z_0| < r$ (the complex derivative). These definitions are the same. If f is bounded (with bound M) in a disk of radius r about z_0, then the Taylor series converges to $f(z)$ as long as $|z - z_0| < r$. Here is one of the proofs you might find in a complex analysis book, but it is not the complex analysis version of the real calculus discussion of Taylor series.

Suppose Γ is a contour that winds once around z_0 with radius r. We rewrite the representation formula (19) as

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\Gamma \frac{f(w)}{w - z_0 + z - z_0} \, dw \\ &= \frac{1}{2\pi i} \oint_\Gamma \frac{f(w)}{w - z_0} \frac{1}{1 - \dfrac{z - z_0}{w - z_0}} \, dw \; . \end{aligned}$$

Define the complex ratio on the right to be

$$q = \frac{z - z_0}{w - z_0} \; .$$

Define $r_1 = |z - z_0| < r$ and $\rho = \frac{r_1}{r} < 1$. Then ($|z - z_0| = r_1$ and $|w - z_0| = r$)

$$|q| \leq \frac{r_1}{r} = \rho < 1 \; .$$

This implies that the geometric series

$$\frac{1}{1-q} = \sum_0^\infty q^n$$

converges absolutely. The finite geometric series is

$$1 + \cdots + q^N = \frac{1-q^{N+1}}{1-q} \ ,$$

so we have an error bound for partial sums:

$$\left| \frac{1}{1-q} - \sum_{n=0}^{n=N} q^n \right| = \frac{\left|q^{N+1}\right|}{|1-q|} \leq \frac{|q|^{N+1}}{1-|q|} = \frac{\rho^{N+1}}{1-\rho} \ . \tag{28}$$

We put the geometric series and bounds into the representation formula and the derivative formulas (23) to get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_\Gamma \frac{f(w)}{w-z_0} \sum_0^\infty \left(\frac{z-z_0}{w-z_0} \right)^n dw \\ &= \sum_0^\infty (z-z_0)^n \, \frac{1}{2\pi i} \oint_\Gamma \frac{f(w)}{(w-z_0)^{n+1}} \\ f(z) &= \sum_0^\infty (z-z_0)^n \, \frac{f^{(n)}(z)}{n!} \ . \end{aligned}$$

We know the sum on the right converges to $f(z)$ because of the remainder bound that we got for geometric series:

$$\left| f(z) - \sum_0^N (z-z_0)^n \, \frac{f^{(n)}(z)}{n!} \right| \leq M \frac{\rho^{N+1}}{1-\rho} \ . \tag{29}$$

One consequence of the convergence of Taylor series is the uniqueness of *analytic continuation*. Suppose $f(z)$ has is defined in a disk or other simple set, Ω_0. An *analytic continuation* of f is a function $g(z)$ defined on a bigger set Ω_1 so that $g(z) = f(z)$ for $z \in \Omega_0$. For example, let Ω_0 be the interior of the unit disk in the complex plane and define

$$f(z) = \sum_0^\infty z^n \ .$$

The function

$$g(z) = \frac{1}{1-z}$$

is an analytic continuation of f because $g(z) = f(z)$ if $|z| < 1$ and $g(z)$ is defined in Ω_1, which is the whole complex plane except the point $z = 1$. The unique continuation theorem says that the analytic continuation, if there is one, is unique. If $h(z)$ and $g(z)$ are analytic functions defined in Ω_1 so that $h(z) = g(z)$ in Ω_0, and if Ω_1 is connected (definition hinted at below), then $h(z) = g(z)$ in Ω_1.

The proof involves looking at the difference $k(z) = g(z) - h(z)$.

8 Poles and residues

An analytic function has a *simple pole* at $z = z_0$ if $f(z)$ is a differentiable function of z in some neighborhood of z_0, but not at z_0, and

$$f(z) = \frac{r}{z - z_0} + g(z) , \tag{30}$$

where $g(z)$ is analytic in a neighborhood of z_0 that includes z_0. The coefficient r is the *residue* of f at z_0. It may be found as the limit

$$r = \text{res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z) .$$

If Γ is a contour that winds around z_0 once in the counter-clockwise direction (a circle or a deformation of a circle), then

$$r = \frac{1}{2\pi i} \oint_\Gamma f(z)\, dz . \tag{31}$$

We prove this formula by taking the limit $\delta \to 0$ and deforming Γ to a circle of radius δ about z_0. The contribution from g vanishes in that limit.

Suppose f is defined and analytic in a neighborhood of z_0 but possibly not at z_0 itself. If f has a simple pole at z_0, then f "blows up" as $z \to z_0$ "like $1/|z - z_0|$". Exactly what it means to "blow up like" something can be different from place to place. For example, we can define

$$M_\delta = \max_{|z - z_0| = \delta} |f(z)| .$$

Then we can ask that there are constants $0 < C_1 < C_2$ so that

$$C_1 \leq \liminf_{\delta \to 0} \delta M_\delta \leq C_2 .$$

Another surprising fact about analytic functions is that if f doesn't blow up at least like $1/|z - z_0|$, then f doesn't blow up at all. Not only is f bounded in a neighborhood of z_0, but there is a value of $f(z_0$ so that the function is analytic at z_0 also. If f blows up slower than $1/|z - z_0|$, then f has a *removable singularity* at z_0. This means that although we thought might be a "singularity" (a point where f is not analytic), we were wrong. If you give the right value of $f(z_0)$ the singularity is "removed" (actually, it was never there, but we can't change terminology).

It is important to get the hypotheses of the removable singularity theorem completely. We require that there is a $\Delta > 0$ so that if $z \neq z_0$ but $|z - z_0| \leq \Delta$ then f is differentiable in the complex sense at z. We also require that

$$\delta M_\delta \to 0 \ , \quad \text{as} \ \ \delta \to 0 \ . \tag{32}$$

This seems to allow $f(z)$ to blow up as $z \to z_0$, but slower than $1/|z - z_0|$. What about the function $f(z) = z^{-\frac{1}{2}}$? This function blows up like $|z - z_0|^{-\frac{1}{2}}$ and seems to satisfy (32). But it's singularity at z_0 is not removable. There is no analytic function defined in a neighborhood of $z = 0$ whose values are $z^{-\frac{1}{2}}$ near zero. The answer is that $z^{-\frac{1}{2}}$ cannot be defined in a continuous way near $z = 0$. There has to be some branch cut where it is discontinuous.

To prove the removable singularity theorem, we show that the Cauchy representation formula (19) holds for this f, where $\Gamma = \Gamma^+_\Delta$, which is the circular contour with radius Δ that goes in the counter-clockwise direction about z_0 (parametrized by $\zeta^+_\Delta(t) = z + 0 + \Delta e^{it}$ for $0 \leq t \leq 2\pi$).

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma^+_\Delta} \tag{33}$$

Of course, z must be inside this circle, which means $|z - z_0| < \Delta$. The hypotheses of the representation formula (19) are not satisfied, because we don't yet know that $f(z)$ is defined or differentiable at z_0. We will show that. Once we do, the theorem is proven. We define $f(z_0)$ to be the value given by (19). We already showed that the function defined in this way is analytic.

The proof of (33) uses a more complicated closed contour

$$\Gamma = \Gamma^+_\Delta + \Gamma_2 + \Gamma^-_\delta + \Gamma_4 \ .$$

By adding contours, we mean that we make the single contour Γ by tracing the contours in the order written. It happens that each contour ends where the next one starts, so together they form a single closed contour. Assume that $\delta < \Delta$. We will get (33) in the limit $\delta \to 0$. The connecting contour goes from the start/end point of $\Gamma^+\Delta$ to the start/end point of Γ^-_δ. It is parameterized by $\zeta_2(t) = (1-t))(z_0 + \Delta) + t(z_0 + \delta)$, for $0 \leq t \leq 1$. The inner contour is a circle of radius δ in the clockwise direction (the "minus" direction), parametrized by $\zeta^-_\delta(t) = z_0 + \delta e^{-it}$ for $0 \leq t \leq 2\pi$.

We call the circular contours Γ^+_Δ, and Γ^-_δ. The straight line parts are Γ_2 and Γ_4, parameterized by $\zeta_2(t) = (1-t)\Delta + t\delta$ and $\zeta_4(t) = (1-t)\delta + t\Delta$. The point of this contour is that it wraps around any point z with $\delta < |z - z_0| < \Delta$ in the counter-clockwise direction (if z is not on Γ_2). You should draw the picture to show that this four part contour may be deformed to a small circle about z. The point of wrapping around z is that we have the Cauchy integral representation

$$f(z) = \frac{1}{2\pi i} \left[\int_{\Gamma^+_\Delta} \frac{f(w)}{w - z} \, dw \right]$$

The last contour is the reverse of Γ_2. It is a straight line segment from the inner circle to the outer circle parametrized by $\zeta_4(t) = (1-t)(z_0+\delta) + t(z_0+\Delta)$. You integrate over Γ by adding the integrals over the pieces

$$\oint_{\Gamma} g(w)\,dw = \oint_{\Gamma_\Delta^+} g(w)\,dw + \int_{\Gamma_2} g(w)\,dw + \oint_{\Gamma_\delta^-} g(w)\,dw + \int_{\Gamma_4} g(w)\,dw \; .$$

The closed contour integrals are written with $\oint$ and the non-closed ones with $\int$.

We need only add them up and take the limit $\delta \to 0$. First,

$$\frac{1}{2\pi i}\oint_{\Gamma} \frac{f(w)}{w-z}\,dw = f(z) \; .$$

That's the Cauchy integral formula (19). There are some details: if z is on Γ_2, we need to connect the inner and outer circles with a different path. You have to check by drawing pictures (see also an exercise) that it is possible to deform Γ to a small circle about z. The integrals over Γ_2 and Γ_4 cancel because they integrate the same function in the opposite direction. The hypotheses (32) implies that the inner integral vanishes in the limit $\delta \to 0$. In fact

$$\begin{aligned}\left|\frac{1}{2\pi i}\oint_{\Gamma_\delta^-} \frac{f(w)}{w-z}\,dw\right| &\leq \frac{1}{2\pi}\left|\Gamma_\delta^-\right| \max\left|\frac{f(w)}{w-z}\right| \\ &\leq \delta\frac{M_\delta}{\min|z-w|}\end{aligned}$$

The min in the denominator is over the circle Γ_δ^-, which is shrinking to the point z_0. Therefore it converges to $|z-z_0|$ as $\delta \to 0$. If $\delta M_\delta \to 0$ (the hypothesis) then the Γ_δ^- integral converges to zero. This is the proof of the removable singularity theorem.

With the technical theorem out of the way, we can get to the important thing – poles and residues. Suppose $f(z)$ is differentiable in a neighborhood of z_0 but "blows up" (has an actual singularity) as $z \to z_0$. We saw that the singularity is removable unless $\limsup_{\delta\to 0} \delta M_\delta > 0$. Therefore, we ask what f can look like near z_0 if there is a C with

$$|f(z)| \leq C\,|z-z_0| \; .$$

This is easy because $g(z) = (z-z_0)f(z)$ has a removable singularity. Therefore $g(z)$ is analytic (has a convergent Taylor series) in a neighborhood of z_0. We write

$$\begin{aligned}(z-z_0)f(z) &= \sum_0^\infty a_n(z-z_0)^n \\ &= a_0 + (z-z_0)\sum_1^\infty a_n(z-z_0)^{n-1} \; .\end{aligned}$$

The sum on the right converges and defines an analytic function

$$h(z) = \sum_0^\infty a_{n+1}(z-z_0)^n \; .$$

This shows that

$$f(z) = \frac{a_0}{z-z_0} + h(z) \; . \tag{34}$$

If Γ is a closed contour that winds once in the counter-clockwise direction about z_0 and stays in the region where

$$h(z) = (z-z_0)f(z) - \frac{a_0}{z-z_0}$$

is analytic, then

$$\oint_\Gamma f(z)\,dz = 2\pi i a_0 \quad , \qquad \text{where } \; a_0 = \lim_{z\to z_0}(z-z_0)f(z) \; . \tag{35}$$

A function that may be written as (34) is said to have a *simple pole* at z_0. The number a_0 defined in (35) is the *residue* of the pole at z_0.

It is possible for f to blow up in a more complicated way at z_0. Suppose, for example, that

$$|f(z)| \leq C\left|z-z_0\right|^2 \; .$$

Then $(z-z_0)^2 f(z)$ has a removable singularity, so

$$f(z) = \frac{a_0}{(z-z_0)^2} + \frac{a_1}{z-z_0} + h(z) \; , \tag{36}$$

where h is analytic in a neighborhood of z_0. Here,

$$\begin{aligned} a_0 &= \lim_{z\to z_0}(z-z_0)^2 f(z) \\ a_1 &= \lim_{z\to z_0}(z-z_0)\left(f(z) - \frac{a_0}{(z-z_0)^2}\right) \; . \end{aligned}$$

It may not seem obvious that the a_1 limit exists, but we have proven that it does. Corresponding to (35) we have

$$\oint_\Gamma f(z)\,dz = 2\pi i a_1 \; .$$

A function that satisfies (36) has a *double pole*, or a pole *of degree* 2, at z_0. We see that the residue from a closed contour integral about z_0 depends not on the *leading order term*, which would be a_0, but on the next term. The leading order term contributes zero because

$$\oint_\Gamma \frac{1}{(z-z_0)^2}\,dz = 0 \; .$$

Clearly, there can be poles of any positive integer degree. If

$$|f(z)| \leq C\,|z-z_0|^n$$

then $f(z)$ has a pole of degree at most n. If f satisfies no inequality like this for any n, then f has an *essential singularity* at z_0. The function $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z = 0$.

9 Inverting a Dirichlet series

We practice many of the ideas described above as we derive the formula (3). If we can interchange the Dirichlet sum and the contour integral, we have a sum of contour integrals of the form

$$\int \frac{1}{s}\frac{x^s}{n^s}\,ds\ .$$

We define $y = x/n$, so $y > 1$ is the same as $n < x$. The formula (3) seems to follow from

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{y^s}{s}ds \;=\; \begin{cases} 0 & \text{if } y < 1\ , \\ 1 & \text{if } y > 1\ . \end{cases} \tag{37}$$

We show that this is true if $\sigma > 0$. "Show" is not exactly what we do. Instead, this is an outline, and you are supposed to fill in the details and write out a full proof. The steps that need filling in are marked in **bold**.

The contour is the vertical line in $\mathbb{C}$ with fixed real part σ and all t. The contour may be parametrized by $\zeta(t) = \sigma + it$, with $-\infty \leq t \leq \infty$. The corresponding differential is $ds = idt$ The parametrized integral is

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{y^{\sigma+it}}{\sigma+it}\,idt\ .$$

This is an improper integral, so the definition is

$$\lim_{R\to\infty}\int_{\sigma-iR}^{\sigma+iR}\frac{y^s}{s}\,ds\ . \tag{38}$$

Step 1: Prove that the numbers (38) form a Cauchy sequence as $R \to \infty$. To do this, find an expression M_R with $M_R \to 0$ as $R \to \infty$ and, if $R' > R$, then

$$\int_R^{R'}\frac{y^s}{s}\,ds \leq M_R\ .$$

For this, write $y^s = y^\sigma e^{it\log(y)}$ and take y^σ outside the integral. You are left with

$$\int_R^{R'}\frac{e^{it\log(y)}}{\sigma+it}\,dt\ .$$

This integral (with $R' = \infty$) does not converge absolutely, because the denominator goes to zero only like $1/t$. However, the factor $1/(\sigma + it)$ goes to zero very smoothly, and is nearly constant over each period of the oscillatory function $e^{it\log(y)}$. (Oscillatory only if $y \neq 1$, and we are interested in $y > 1$ or $0 < y < 1$.) Here is a trick for finding cancellation from oscillatory integrals which I learned from a paper by the amazing (and difficult) Swedish mathematician Hörmander. Note that

$$\frac{d}{dt} e^{it\log(y)} = i\log(y)\, e^{it\log(y)} \; .$$

Therefore

$$e^{it\log(y)} = \frac{1}{i\log(y)} \frac{d}{dt} e^{it\log(y)} \; .$$

We can integrate by parts to get (figure out the sign)

$$\begin{aligned}
\int_R^{R'} \frac{e^{it\log(y)}}{\sigma + it}\, dt &= \frac{1}{i\log(y)} \int_R^{R'} \left[\frac{d}{dt} e^{it\log(y)} \right] \frac{1}{\sigma + it}\, dt \\
&= \frac{1}{i\log(y)} \left[\frac{e^{iR'}\log(y)}{\sigma + iR'} - \frac{e^{iR}\log(y)}{\sigma + iR} \right] \\
&\pm \; \frac{1}{i\log(y)} \int_R^{R'} e^{it\log(y)} \frac{1}{(\sigma + it)^2}\, dt \; .
\end{aligned}$$

The last integral does converge absolutely as $R' \to \infty$ for fixed R.

Now suppose $y > 1$ and consider the closed contour $\Gamma_{R,L}$, that consists of 4 pieces. The first piece is the integral (38). We will close this contour using a box built on the left of this vertical piece. The top of the box is Γ_2 that makes a horizontal line from $\sigma + iR$ to $-L + iR$, with $L > 0$ that will soon go to infinity. The left side of the box, Γ_3, is a vertical segment from $-L + iR$ to $-L - iR$. This may be parameterized by $\zeta_3(t) = -L - it$, for $-R \leq t \leq R$ (moving down from $-L + iR$ to $-L - iR$. The last part piece, Γ_4, is a horizontal segment from $-L - iR$ to $\sigma - iL$.

Step 2: Show that

$$\int_{\Gamma_3} \frac{y^s}{s}\, ds \to 0 \quad , \quad \text{as} \;\; L \to \infty \; .$$

This has to do with the behavior of y^{-L} as $L \to \infty$ if $y > 1$.

With this we can take $L = \infty$, which really means that Γ_2 goes from $\sigma^i R$ to $-\infty + iR$.

Step 3: Show that

$$\int_\sigma^{-\infty} \frac{y^{u+iR}}{u + iR}\, du \to 0 \quad , \quad \text{as} \;\; R \to \infty \; .$$

You can pull y^{iL} out of the integral. You can write $y^u = e^{u\log(y)}$ and note that $\log(y) > 0$ while $u \to -\infty$. Also, over all of Γ_2, we have $|1/(u + iR)| \leq 1/R$

(why?).

Step 4: Show that

$$\frac{1}{2\pi i}\int_{\Gamma_{R,L}} \frac{y^s}{s}\, ds = 1 \ .$$

This is true for any $y > 0$, but it is irrelevant for (37) unless $y > 1$ because steps 2 and 3 don't work.
Step 5: Show that if $0 < y < 1$, the closed contour must complete to the

10 Exercises

1. Use the definition of the derivative of functions of a complex variable and the properties of the complex exponential to verify the following differentiation formulas

$$\begin{aligned}
\frac{d}{dz} z^3 &= 3z^2 \\
\frac{d}{dz} z^n &= nz^{n-1} \\
\frac{d}{dz} e^z &= e^z \\
\frac{d}{dz} e^{az} &= ae^{az} \\
\frac{d}{dz} a^z &= \log(a)a^z \quad , \quad \text{if } a > 0 \ .
\end{aligned}$$

2. Use the proofs from basic calculus to prove the same formulas are true in complex calculus. Assume that the functions $f(z)$ and $g(z)$ are analytic

$$\begin{aligned}
\frac{d}{dz}\left(f(z)g(z)\right) &= f'(z)g(z) + f(z)g'(z) \\
\frac{d}{dz} f(g(z)) &= f'(g(z))\, g'(z) \ .
\end{aligned}$$

3. Calculate, using the product rule, the chain rule, and implicit differentiation

$$\begin{aligned}
&\frac{d}{dz} e^{z^2} \\
&\frac{d}{dz}\frac{1}{z} \quad \big(\text{hint: try } f(z) = z, \text{ and } g(z) = z^{-1}.\big)
\end{aligned}$$

4. Write the function $f(z) = z^{-1}$ in the form $f = u + iv$. Find formulas for $u(x, y)$ and $v(x, y)$. Calculate the partial derivatives of these functions and check that they satisfy the Cauchy Riemann equations.

5. Finish the proof of the parametrized integration formula (11). Define the maximum spacings for these to be

$$M_t = \max \Delta t_k \quad , \qquad M_z = \max |\Delta z_k| \ .$$

Show that $M_z \to 0$ as $M_t \to 0$. Take $\delta > 0$ so that if $M_t \leq \delta$ then

$$\left| \sum_{k=0}^{n-1} f(\zeta(t_k))\zeta'(t_k)\Delta t_k - \int_a^b f(\zeta(t))\zeta'(t)\, dt \right| \leq \frac{\epsilon}{3} \ ,$$

and if $M_z \leq \delta$, then something similar for

$$\int_\Gamma f(z)\, dz \ ,$$

and the thing relating Δz_k to Δt_k (with the rectifiability constant L replaced with $b - a$). Add these three inequalities to conclude that

$$\left| \int_\Gamma f(z)\, dz - \int_a^b f(\zeta(t))\zeta'(t)\, dt \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \ .$$

6. Consider the definite integral

$$A = \int_{-1}^{1} x^2\, dx = \frac{2}{3} \ . \tag{39}$$

Consider the following "substitution": $x = e^{i\theta}$ and $dx = ie^{i\theta} d\theta$. If we plug in the new variable, we get an expression for A as an integral over θ. The range is from $\theta = -\pi$, which corresponds to $x = -1$, to $\theta = 0$, which corresponds to $x = 1$:

$$A = \int_{-\pi}^{0} \left(e^{i\theta}\right)^2 ie^{i\theta}\, d\theta \ . \tag{40}$$

Calculate the integral using the facts of calculus (such as $\frac{d}{d\theta} e^{3i\theta} = 3ie^{3i\theta}$). Draw the contours in the complex plane that the integrals (39) and (40) correspond to. Explain how Cauchy's theorem implies that the answers are the same.

7. Calculate

$$\int_{\Gamma_+} z^{-2} dz \ \text{ and } \ \int_{\Gamma_+} z^{-2} dz \ ,$$

and see that they are the same. Integration over the two contours gives the same result (though you're not being asked to do the calculation) for $f(z) = z^n$ for any integer n except $n = -1$.

8. Consider the family of contours Γ_R that go from 0 to π by first going up, then over, then down. That is, Γ_R goes from 0 straight up to iR, then straight over to $iR + \pi$, then straight down to π. Suppose $f(z)$ is an analytic function of z for all z. Write an explicit representation of
$$\int_{\Gamma_R} f(z)\, dz$$
as a sum of ordinary integrals over the three parts. Compute the derivatives of these three parts explicitly to see that the integral is independent of R. You can start with
$$\frac{d}{dR}\int_0^R f(iy) i dy = i f(iR) \ .$$

9. Consider the integral
$$\int_0^\pi e^{ix}\, dx \ .$$
Consider replacing this contour with Γ_R from Exercise 8. Take the limit $R \to \infty$ and show that you are left with
$$\int_0^\pi e^{ix}\, dx = \pm i \int_0^\infty e^{-y}\, dy + (\text{a similar integral}) \ .$$
Calculate the integrals on the left and right and check that they are equal.

10. Show that (27) is sharp in the following example. Let $f(z) = \frac{1}{z}$. Take $z_0 = 1$ and $r < 1$. Show that $M = \frac{1}{1-r}$ (the constant in (25)). Show that if z is real and $1 > z > 1 - r$, then the two sides of (27) are equal.

11. Let $f(z)$ be an analytic (differentiable) function in a neighborhood of z_0 with Taylor series
$$f(z) = \sum_0^\infty a_n (z - z_0)^n$$
that converges to f for $|z - z_0| \le \Delta$. Assume that f is not "locally identically zero", which means that there is a sequence $w_k \to z_0$ as $k \to \infty$ so that $f(w_k) \neq 0$.

    (a) Show that if there is no sequence like w_k, then there is a $\delta > 0$ so that $f(z) = 0$ if $|z - z_0| \le \delta$. (This part is practice with $\epsilon - \delta$ style mathematical analysis.)

    (b) Show that if f is not locally identically zero (there is no sequence like w_k), then there is an n, and a $C > 0$, and a $\delta > 0$ so that
$$|f(z)| \ge C\, |z - z_0|^n \quad \text{if} \quad |z - z_0| \le \delta \ .$$

Hint: If $a_n = 0$ for all n then f is locally identically zero. Therefore, there is a smallest n with $a_n \neq 0$. Define

$$g(z) = \sum_{m>n} a_m \left(z - z_0\right)^m \ ,$$

and show that $|g(z)| \leq B\left|z - z_0\right|^{n+1}$. Choose $\delta = |a_n|/(2B)$ (or something like that).

(c) Show that $1/f(z)$ either is analytic in a neighborhood of z_0 or has a pole of finite order, not an essential singularity.

(d) Find a formula for the residue of $1/f(z)$ at z_0 in the case the pole has order 1 or order 2.

12. In this exercise, suppose $s = \sigma + it$ with $\sigma > \sigma_0 > 0$.

(a) Show that

$$a_n(s) = n^{-s} - \int_n^{n+1} x^{-s}\,dx$$

is analytic and that

$$|a_n(s)| \leq Cn^{-1-\sigma_0} \ .$$

(b) Show that

$$\zeta(s) = \frac{1}{s-1} + r(s) \ , \qquad (41)$$

where

$$r(s) = \sum_1^\infty a_n(s)$$

is an analytic (differentiable) function of s. *Hint*: The theory in the notes shows that r is automatically differentiable if the sum that defines it converges uniformly enough.

(c) Show that there is a function, called $\zeta(s)$, which: (1) is equal to $\sum n^{-s}$ for $\sigma > 1$, (2) is analytic for $\sigma > 0$ except for a pole at $s = 1$. (This is an *analytic continuation* of the zeta function from $\sigma > 1$ to $\sigma > 0$.)

(d) Show that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_1^\infty \Lambda(n)n^{-s}$$

has a simple pole at $s = 1$ with residue 1. Be careful to state this precisely: There is an open set that contains $\sigma > 1$ and a neighborhood of $s = 1$ (including some s values with $\sigma < 1$ for s near 1) and an analytic continuation of ζ'/ζ to that open set, and that analytic continuation has a simple pole, etc. *Hint*: Exercise 11 is useful.

13. Carry out the steps 1-5 to prove the formula (37). Write out a more or less complete proof. This is a writing exercise and a math exercise. Most math exercises have fewer steps. Part of the challenge is organization.