Final Exam

Rules: This is due either in my office or in my box by 6pm Tuesday, May 16. You may use posted class notes and textbooks but you may not consult with others on the questions. Some of the questions are meant to be more challenging than others. Just do your best.

Questions

1. Explain the structure of the multiplicative group of the integers mod 18 relatively prime to 18. How many elements does it have? What is its cycle structure? Give generators for the cycles.

2. Assume Mertens' theorem

$$\lim_{n\to\infty}\sum_{p\leq x}\frac{1}{p} = \log(\log(x)) + O(1) \quad \text{as} \quad n\to\infty \ .$$

   Show that if

$$\pi(x) = \frac{Lx}{\log(x)} + O\left(\frac{x}{\log^2(x)}\right) \quad \text{as} \quad x\to\infty \ ,$$

   then $L = 1$. *Hint*: Use Abel summation to show that Mertens' theorem is a consequence of the prime number theorem.

3. If $f(x)$ is a function of the integer x that is periodic with period n, then the DFT is the unique function $\widehat{f}(k)$ that satisfies

$$f(x) = \sum_{k=0}^{n-1} e^{2\pi ikx/n}\widehat{f}(k) \ .$$

   Suppose $n = pq$ (p and q do not have to be prime). Prove the discrete Poisson summation formula

$$\sum_{j=0}^{q-1} f(jp) = C_{p,q}\sum_{k=0}^{p-1}\widehat{f}(kq) \ .$$

   Find a formula for $C_{p,q}$.

4. Use the Riemann functional equation and the formula

$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

to find the value for $\zeta(-1)$. It is amusing, but not rigorous (to say the least) to write this as

$$\zeta(-1) = \sum_{1}^{\infty} n \, .$$

5. Let S_q be the set of prime numbers that are squares mod q (q being a fixed prime). Let T_q be the set of prime numbers that are not squares mod q and let N_q be the set of positive integers that are squares mod q. Define

$$\zeta_q(s) = \sum_{n \in N_q} n^{-s} \, .$$

   (a) Show that the sum converges absolutely if and only if $\mathrm{Re}(s) > 1$.

   (b) Show that

$$\zeta_q(s) = \frac{C}{s-1} + O(1) \quad \text{as} \quad s \to 1 \, ,$$

   and identify the constant C.

   (c) Prove the product formula, which holds if $\mathrm{Re}(s) > 1$:

$$\zeta_q(s) = \left(\prod_{p \in S_q} \frac{1}{1 - p^{-s}} \right) \left(\prod_{r \in T_q} \frac{1}{1 - r^{-2s}} \right) \, .$$

6. Use complex contour integration to evaluate the Fourier transform of

$$f(x) = \frac{1}{1 + x^2} \, .$$

We did this very quickly and incompletely in class. Explain the completion of the contour to make a closed contour. Note that the completion depends on the sign of ξ (the Fourier transform variable). State the residue theorem you use to evaluate the integral around the closed contour. *Hint:* The answer has the form

$$\widehat{f}(\xi) = a e^{-b|\xi|} \, .$$

All the complex analysis will identify the numbers a and b. Use this to find an explicit formula for

$$\sum_{-\infty}^{\infty} \frac{1}{1 + n^2} \, .$$