Analytic Number Theory, undergrad, Courant Institute, Spring 2017
http://www.math.nyu.edu/faculty/goodman/teaching/NumberTheory2017/index.html

Section 1, Introduction

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1 Preface.

These are informal notes for an upper level undergraduate class on analytic number theory. I hope the class and these notes will give students some experience with mathematics the way mathematicians think about it. I will cover some of the early big theorems in the subject that involve the Riemann zeta function and its relatives, and the Poisson summation formula. Depending on time and interest, I may be able to cover some material on partitions using ideas derived from Hardy and Ramanujan.

Analytic number theory is a living subject. Ancient Greek mathematicians had studied prime numbers and proved that there are infinitely many of them. About two thousand years later, Euler proved that $\sum p^{-1}$ is infinite (the sum is over all prime numbers). For this, Euler used analysis – in particular an infinite product formula for what now is called the *Riemann zeta function*. Soon after that people started to believe what is now called the *prime number theorem*, which states that the density of prime numbers around x is approximately $1/\log(x)$. By about 1840, Dirichlet proved that $\sum p^{-1}$ is infinite even if you sum only over primes, for example, equal to 3 mod 11. His proof used more product formulas, plus what now are called the Dirichlet characters. Twenty years later, Riemann used the zeta function as a function of the complex variable s (going far beyond Euler and Dirichlet) to outline a proof of the prime number theorem. The full proof using Riemann's ideas was completed around 1895, but only after the subject of complex analysis had been developed. It turned out that the zeta function is very interesting in itself. For example, it was shown about 20 years ago that the zeros of the zeta function (number ρ with $\zeta(\rho) = 0$) have much in common with the eigenvalues of random matrices. One of the most famous unsolved problems in mathematics is the proof that all these zeros (except the "trivial" ones) have the form $\rho = \frac{1}{2} + it$ (with t real). Erdös and Selberg, in the late 1940's, found a different proof of the prime number theorem that does not use the zeta function of complex analysis. The Selberg *large sieve* method is the basis for the proof in 2013 by unknown mathematician Yitang Zhang of the *bounded gap* theorem: There is a number A so that there are infinitely many distinct prime number pairs $|p - p'| \leq A$. The best known A today is a few hundred. People believe the *twin prime* conjecture, that there are infinitely many pairs with $A = 2$, such as 5,7, and 11,13, and 137,139, etc.

These achievements rely on many beautiful ideas and surprising formulas. There are the infinite product formulas of Euler and Dirichlet described below. Other infinite product formulas are described in Assignment 1. The *Poisson* *summation formula*, roughly $\sum f(n) = \sum \widehat{f}(k)$, relates the values of a function f to the values of its Fourier transform, $\widehat{f}$. We will use this to derive $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$ and to study the Gauss lattice point problem: how many integer pairs (i,j) are there with $i^2 + j^2 < r^2$. We will see that this number is much closer to πr^2 than would seem possible. The list of surprising and powerful formulas is very long. Hopefully, the class will get to the *Rogers Ramanujan* identities.

The course prerequisites, beyond the calculus sequence, are linear algebra and "ϵ δ" analysis. I assume the student has understood these subjects but may not yet be facile with them. Some arguments will be given in a way experienced mathematicians will find pedantic. In my experience, this is appropriate for bright undergraduates.

I do not assume that the student has learned complex analysis, so a certain amount of class time and note space will be devoted to the basics of contour integration. Much of complex analysis was developed in order to do analytic number theory. The proofs will often be different from the ones in introductory analysis. The proofs here depend on interesting formulas and estimates.[1] The estimates for analytic number theory often involve surprising insights that come from algebra. Abstract algebra and basic number theory also are not prerequisites for this class. The parts of algebra we need will be explained as they arise.

I believe an undergraduate class should be structured differently from a graduate class. One difference is that it is impractical to assume too much specific background from undergraduates. More importantly, an undergraduate class is not necessarily professional training for future academic mathematicians. I will not assume that the student brings this motivation with her/him. On the contrary, I will try to explain why I find things interesting. I will not discuss subjects exhaustively, but try to pick interesting nuggets that could form the motivation for a more systematic graduate class on analytic number theory. I will dwell on ideas and mathematical techniques that are applicable beyond the number theory discipline. Quoting from the preface of the book of Hardy and Wright listed below: "We may have succeeded at the price of too much eccentricity, or we may have failed. But we can hardly have failed completely, the subject-matter being so attractive that only extravagant incompetence could make it dull."

This course has two models. One is an undergraduate class based on the textbooks of Apostol or Jameson (see references below). While these are excellent books that could form the basis for an excellent class. But I want something that is both less slick and faster paced. As far as possible, I want to put the number theory first and fill in the analytical background only as needed and when it is needed. This way, students should know why they are doing what they're doing and what is important about it. A more traditional course may give the students more systematic preparation, but gives a misleading picture

[1] An *estimate* is an inequality that gives a useful bound on the size of a sum or integral. An estimate, in this sense, is not a guess at the true value. One estimate is *sharper* than another if it is closer to the actual value.

of mathematics. I find Apostol or Jameson frustrating precisely because it isn't clear where they're going, or even when the proof is over where they've been

The other model is the lecture notes of Elkes. Those are at a higher mathematical level than a typical upper division undergraduate class. Also, they have more digressions and explorations that I think this class has time for. On the positive side, Elkes is a real master of the subject who has thought a lot about why things are as they are. Mathematicians have a habit of undisciplined exploration, which is how they discover stuff, but would mean that we don't cover as much as I hope to cover. I will refrain (with effort on my part) from examining each theorem to find the weakest hypotheses or most general setting. That may not be a good thing.

You have to be an *active learner* to understand mathematics at this level. You must challenge yourself by making up exercises and questions as you read every paragraph. If I don't stop to ask what would happen if you weaken a hypothesis, you should. Maybe you can find an interesting example or counter-example. But even if you don't, you will understand the issue better. I have refrained from putting such "routine" (not necessarily easy) exercises in the homework. I want students to develop the "mathematical maturity" to come up with those independently.

2 References.

If the class notes get too sketchy or confusing, or just out of curiosity, people may want to consult other sources. Here are some suggestions

Number theory

Harold Stark, *An Introduction to Number Theory.* An elementary and reader friendly introduction to basic number theory.

Hardy and Wright, *Theory of Numbers.* This is the obviously best choice if need to look up something about basic number theory such as prime factorization or modular arithmetic.

Serre, *A Course of Arithmetic.* Beautiful elegant and simple proofs, at a much more advanced level.

Analytic number theory

Apostol, *Analytic Number Theory.* This is considered a good book for advanced undergraduates, but I find it almost unreadable. Everything is explained carefully, but the overall motivation and structure is almost entirely missing. There is lemma after lemma, which are very hard to remember if you don't know what they're for, or what would happen without them. Comparing to Serre or Elkes, you can see how unmotivated and inefficient this is.

Jameson, *The Prime Number Theorem.* Similar to Apostol.

Elkes, *Introduction to Analytic Number Theory.* This is full of motivation and explanation not only of what the proofs are, but why they work. It's pitched at a higher level, so some of what he thinks is obvious may be far from obvious. There are things that I can't understand either. If you do understand it, it will be very clear to you.

Complex analysis There are many quite similar books. You should look at a few to see which is most helpful to you. Here are my favorites:

Marsden and Hoffman, *Basic Complex Analysis.* An easy undergraduate level text. It's not the most efficient introduction, particularly if you don't need a review of basic analysis. The Cauchy integral theorem starts on page 95.

Brown and Churchill, *Complex Variables and Applications.* More concrete that Marsden and Hoffman, but still slow.

Alfors, *Complex Analysis.* Harder and faster than the others, but very elegant, insightful, and efficient.