Practice questions for the second quiz

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Rules: Same as for the first quiz.
Hints

- These questions may be harder than the actual quiz questions will be.
- The actual quiz questions may be related to these.
- Consider the exercises in the Complex Variables section of notes also as sample questions.

Questions

1. We saw that the real function of a real variable x, $f(x) = e^{-\frac{1}{x^2}}$ is differentiable for $x \neq 0$, and if you take $f(0) = 0$ then f is differentiable at $x = 0$ also. Show that the complex function of the complex variable z, $f(z) = e^{-\frac{1}{z^2}}$ is differentiable for all $z \neq 0$. If you define $f(0) = 0$, then $f(z)$ is not differentiable or even continuous or even bounded in a neighborhood of $z = 0$ in $\mathbb{C}$.

2. Suppose $a_n(z)$ is a family of analytic functions defined for $|z| < 1$. Define

$$M_n = \sum_{|z|<1} |1 - a_n(z)| \ .$$

Suppose that

$$\sum_1^\infty M_n < \infty \ .$$

Consider

$$f(z) = \prod_1^\infty a_n(z) \ .$$

Show that this defines an analytic function of z for $|z| < 1$.

3. Suppose that $f(z)$ is an analytic function of z for $|z| < 2$, except possibly at $z = 0$. Suppose that

$$|f(z)| \leq \frac{C}{|z|} \ . \tag{1}$$

and that

$$R = \frac{1}{2\pi 2} \oint_{|z|=1} f(z)\, dz$$

is real. Let x be a real variable. Show that there is a D so that

$$|\mathrm{Re}(f(z))| \le D \ , \quad \text{for} \ \ |x| < 1 \ . \tag{2}$$

Find an example of a function $f(z)$ analytic for $|z| \le 2$ that does not satisfy either (1) or (2).

4. Consider the integral

$$T(k) = \int_{-\infty}^{\infty} e^{-ikx} e^{-x^4} \, dx \ .$$

Show that there is a C with

$$|T(k)| \le C e^{-k} \ .$$

Hint: make it a contour integral in $\mathbb{C}$ and move the contour off the real axis to the contour with $\mathrm{Im}(z) = -1$. Justify this.

5. Suppose $f(s)$ is an analytic function of the complex variable s in a neighborhood of $s = 0$. Find a formula for

$$R = \frac{1}{2\pi i} \oint_{|z|=a} \frac{f(s)}{s^2} \, ds \ .$$

The formula will be in terms of $f(0)$, $f'(0)$, and things like that.

6. Find a formula for

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^s}{s^2} \, ds$$

as a function of the real variable $x > 0$, for $x \ne 1$.

7. Show that if $f(z)$ is analytic for $0 < |z| < 1$ and if

$$|f(z)| \le \frac{C}{|z|^{1.5}} \ ,$$

then

$$|f(z)| \le \frac{D}{|z|} \ .$$

8. Suppose $z = re^{i\theta}$ with $r \ge 0$ and $-\pi < \theta \le \pi$. Define $w(z) = r^{\frac{1}{2}} e^{\frac{i\theta}{2}}$.

   (a) Show that $w(z)$ is an analytic function of z as long as $\mathrm{Im}(z) \ne 0$ or $\mathrm{Im}(z) = 0$ and $\mathrm{Re}(z) > 0$.

   (b) Show that $w^2 = z$ for all z.

   (c) Show that there is no analytic function of z that satisfies $f(z)^2 = z$ defined in a full neighborhood of $z = 0$. (*Hint:* What can you say about the size of the derivative of an analytic function?)

9. Show that the sum

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{1}^{\infty} \Lambda(n) n^{-s}$$

converges absolutely to an analytic function of s if $\sigma = \mathrm{Re}(s) > 1$.