Practice questions for the first quiz

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Rules

- Write all answers in a bluebook (bluebooks provided by the instructor).
- Please explain an answer with both words and mathematical notation. Leaving out one or the other makes it hard to understand your thinking. You should be grammatical, though complete sentences are not required.
- You don't have to answer the questions in order, but please label which question you're answering.
- As far as possible in a test situation, try to write in a way that is easy for an elderly professor with less than perfect eyesight to read – dark pen or pencil, large letters, clear lines, etc.
- Everything you write will be graded unless it is crossed out. If you write a correct and incorrect answer, the negative points for the incorrect answer will cancel the positive points for the correct answer. Please cross out anything you think is wrong.
- You will get 25% credit for a blank answer to any question. You can lose this if you give an incorrect answer.
- You are allowed one *cheat sheet*, which is an 8.5×11 inch piece of paper that you prepare in advance with any information you care to put on it. You are not allowed to use any other materials. Cell phones and watches much not be visible during the quiz.

Hints

- The solutions presented are not the only ones possible.
- I suggest that you work the problems without looking at the solutions.
- On the real quiz, the sentences in your answer may not come out in order. Your answer may be less organized than mine. Just concentrate on getting everything needed to answer the question down. In homework you would then copy the sentences over in order, which you can't do in a quiz.
- The questions are a combination of memory of definitions and theorems (rare for a math quiz) and simple applications (more common).
- The questions focus more than I would like on technical details rather than the big ideas.

- There are more sample questions here than there will be real questions on the quiz. Some of the quiz questions may be easier than these.

Questions

1. Prove the inequality

$$\left| n^{-s} - \int_n^{n+1} x^{-s}\, ds \right| \leq s n^{-(s+1)} \; .$$

Possible solution: The intermediate value theorem says that if $x > n$ then $f(x) - f(n) = f'(\xi)(x-n)$ for some $\xi \in [n,x]$. The hypothesis is that f is differentiable and the derivative is continuous in the interval $[n,x]$. Take $f(x) = x^{-s}$ with $f'(x) = -sx^{-(s+1)}$. This is continuously differentiable for $x \geq 1$, which covers all the values in our problem. Our $|f'|$ is a monotone decreasing function of x for $x \geq 1$, therefore $|f'(\xi)| \leq |f'(n)|$ if $\xi \in [n,x]$. This implies that $|f(x) - f(n)| \leq |f'(n)| = sn^{-s}$. Here, we used the intermediate value theorem and the fact that $|x-n| \leq 1$. Therefore

$$\begin{aligned}
\left| n^{-s} - \int_n^{n+1} x^{-s}\, dx \right| &= \left| \int_n^{n+1} \left(n^{-s} - x^{-s} \right)\, dx \right| \\
&\leq \int_n^{n+1} \left| n^{-s} - x^{-s} \right|\, dx \\
&\leq \int_n^{n+1} s n^{-(n+1)}\, dx \\
&= s n^{-(s+1)} \; .
\end{aligned}$$

2. State the dominated convergence theorem for differentiating an infinite sum and apply the theorem to prove that if $s > 0$ then

$$\frac{d}{ds} \sum_1^\infty e^{-n^2 s} = - \sum_1^\infty n^2 e^{-n^2 s} \; .$$

Possible solution: The dominated convergence theorem for differentiating sums may be stated as follows. $f_n(s)$ is a sequence of functions, defined for $|s - s_0| \leq \delta > 0$. There are a_n with $|f_n(s)| \leq a_n$ for all n and $|s - s_0| \leq \delta$, and $\sum a_n < \infty$. There are b_n with $|f'(s)| \leq b_n$ for all n and $|s - s_0| \leq \delta$. f is differentiable with a continuous derivative for $|s - s_0| \leq \delta$. The sequences a_n and b_n do not depend on s. $\sum b_n < \infty$. The theorem says that

$$F(s) = \sum_1^\infty f_n(s)$$

is defined for all $|s - s_0| \leq \delta$ and that

$$F'(s) = \sum_1^\infty f_n'(s)$$

Our problem has $f_n(s) = e^{-n^2 s}$. For any $s_0 > 0$, define $\delta = s_0/2$. Then $e^{-n^2 s} \leq e^{-n^2(s_0-\delta)} \leq e^{-n(s_0-\delta)}$. This is because $n^2 \geq n$ if $n \geq 1$, as it is here. If we define $r = e^{-n(s_0-\delta)}$, then $0 \leq r < 1$. We take $a_n = r^n$ and $\sum a_n = \sum r^n = 1/(1-r) < \infty$. (You would get full credit if you just say that the geometric series has a finite sum). The b_n need to bound the derivatives: $|f_n'(s)| = n^2 e^{n^2 s} \leq b_n$ whenever $|s - s_0| \leq \delta$. We just showed that for s in this range, $e^{-n^2 s} \leq r^n$. Therefore we can take $n_n = n^2 r^n$ and know that $|f'(s)| \leq b_n$ as desired. The sum that needs to be finite is $\sum b_n = \sum n^2 r^{-n}$. You would get full credit for saying: "since exponentials beat powers of n, this sum is finite." You would lose a little credit for saying: "I'm pretty sure this sum is finite."

There are several ways to prove the sum $\sum n^2 r^{-n}$ is finite. One is to introduce $t = \sqrt{r} < 1$. We know (because exponentials beat powers) that

$$\max_n n^2 t^n = M < \infty\ .$$

Therefore, since $r^n = t^n t^n$, we can calculate

$$\sum n^2 r^n = \sum \left(n^2 t^n\right) t^n \leq M \sum t^n = \frac{M}{1-t} < \infty\ .$$

You don't have to use the square root, any t with $r < t < 1$ would allow this trick, with

$$M = \max_n n^2 \left(\frac{r}{t}\right)^{-n} < \infty\ .$$

This is because $r/t < 1$ (why?). Another approach is to find a formula for the sum. This may be done using a well known trick based on

$$\frac{d}{dr} r^n = n r^{n-1} \quad , \quad r\frac{1}{sr} r^n = n r^n\ .$$

$$\sum_1^N n r^n = r\frac{d}{dr}\sum r^n = r\frac{d}{dr}\frac{1 - r^{N+1}}{1-r}\ .$$

If you calculate the right side, it is clear that it has a limit as $N \to \infty$. This proves that the sum $\sum n r^n$ is finite. The sum $\sum n^2 r^n$ can be done using this trick twice.

3. Use the heuristic density of primes to write an integral approximation to the sum

$$S(x) = \sum_{p \leq x} p\ .$$

Approximate the integral to find an algebraic formula like $\pi(x) \approx \frac{x}{\log(x)}$ for large x.

Possible solution: The heuristic density of primes is

$$\rho(x) = \frac{1}{\log(x)} \; .$$

If the probability that a given n is prime is $\rho(n)$, then $S(x)$ is approximately

$$\int^x y\rho(y)\, dy = \int^x \frac{y}{\log(y)}\, dy \; .$$

Since $\frac{1}{\log(y)} \approx \frac{1}{\log(x)}$ unless y is much smaller than x, we replace $\frac{1}{\log(y)}$ with $\frac{1}{\log(x)}$ in the integral. The result is

$$S(x) \approx \frac{1}{\log(x)} \int^x y\, dy \approx \frac{x^2}{2\log(x)} \qquad \text{(NYU prime sum conjecture)} \; .$$

This would get you most of the points for the problem. You're not being asked to prove the NYU prime sum conjecture, but it is possible to justify the integral approximation using integration by parts (or another way):

$$\begin{aligned} \mathrm{Si}(x) = \int_2^x \frac{y}{\log(y)}\, dy &= \int_2^x \frac{1}{\log(y)} \left(\frac{d}{dy} \frac{1}{2} y^2 \right) dy \\ &= \frac{x^2}{2\log(x)} + \frac{1}{2} \int_2^x \frac{y}{\log(y)^2}\, dy + O(1) \end{aligned}$$

The second integral is smaller than the first roughly by a factor of $\frac{1}{\log(x)}$. You can prove this by splitting the integral into two parts, the first part which is small and the second part where $\frac{1}{\log(y)}$ is small.

$$\begin{aligned} \int_2^x \frac{y}{\log(y)^2}\, dy &= \int_2^{\epsilon x} \frac{y}{\log(y)^2}\, dy + \int_{\epsilon x}^x \frac{y}{\log(y)^2}\, dy \\ &\leq \frac{\epsilon^2 x^2}{2} + \frac{1}{\log(\epsilon x)} \int_{\epsilon x}^x \frac{y}{\log(y)}\, dy \\ &\leq \frac{\epsilon^2 x^2}{2} + \frac{1}{\log(\epsilon x)} \mathrm{Si}(x) \end{aligned}$$

Now choose $\epsilon \to 0$ as $x \to \infty$ so that $\epsilon^2 x^2 << \frac{x^2}{2\log(x)}$ and $\log(\epsilon x) \to \infty$ as $x \to \infty$. This proves that

$$\int_2^x \frac{y}{\log(y)^2}\, dy = o(\mathrm{Si}(x)) \; .$$

4. Show that the following product converges

$$\prod_2^\infty \left(1 + \frac{(-1)^n}{n} \right)$$

Possible solution: The product does not converge absolutely because $\sum |(-1)^n/n| = \infty$. However[1] The product may be written

$$\prod_1^\infty \left[\left(1 + \frac{1}{2n}\right)\left(1 + \frac{-1}{2n+1}\right) \right] .$$

Calculate:

$$\begin{aligned} \left(1 + \frac{1}{2n}\right)\left(1 + \frac{-1}{2n+1}\right) &= 1 + \frac{1}{2n} - \frac{1}{2n+1} + \frac{1}{2n(2n+1)} \\ &= 1 + \frac{(2n+1) - 2n}{2n(2n+1)} + \frac{1}{2n(2n+1)} \\ &= 1 + \frac{1}{2n(2n+1)} + \frac{1}{2n(2n+1)} \\ &= 1 + \frac{2}{2n(2n+1)} \\ &= 1 + O\!\left(\frac{1}{n^2}\right) . \end{aligned}$$

Therefore, the product

$$\prod_1^\infty a_n \quad , \qquad a_n = \left(1 + \frac{1}{2n}\right)\left(1 + \frac{-1}{2n+1}\right)$$

converges absolutely.

Be prepared to supply proofs of the following minor points if asked:

- Justify $\dfrac{2}{2n(2n+1)} = O\!\left(\dfrac{1}{n^2}\right)$
- If

  $$\begin{aligned} P_N &= \prod_{n \le N} \left(1 + \frac{(-1)^n}{n}\right) \\ Q_N &= \left[\left(1 + \frac{1}{2n}\right)\left(1 + \frac{-1}{2n+1}\right) \right] \end{aligned}$$

  then

  $$\lim_{N\to\infty} P_N = \lim_{N\to\infty} Q_N .$$

  For this, it is clear that $P_{2n} = Q_N$, so if $\lim Q_N$ exists (which it does because it converges absolutely), then $\lim P_{2N}$ exists. If $\lim P_{2N}$ exists and $\lim |P_{2N} - P_{2N+1}| = 0$, then $\lim P_N$ exists. Justify $\lim |P_{2N} - P_{2N+1}| = 0$.

[1] The word *however* often signals that a mathematical trick is coming.

5. Write an Euler product representation for the sum

$$f(s) = \sum_0^\infty (2n+1)^{-s} = 1^{-s} + 3^{-s} + 5^{-s} + \cdots \ .$$

Possible solution: The sum is over all odd integers. An odd integer is one that has no 2 in its prime factorization. The Euler product will be over all primes except 2:

$$f(s) = \prod_{p \geq 3} \left(1 - p^{-s}\right)^{-1} \ .$$

The formula is true for all $s > 1$.

6. Let $\zeta_K(s)$ be the zeta function for the Gaussian integers

$$\zeta(s) = \sum_{a \geq 1} \sum_{b \geq 0} \left(a^2 + b^2\right)^{s/2} \ .$$

   (a) Show that the sum converges absolutely for $s > 2$ and that $\zeta_K(s) \to \infty$ as $s \downarrow 2$.

   (b) Assume that there is an Euler product representation for $\zeta_K(s)$ with a product over all Gaussian primes. Show that

$$\sum_p \frac{1}{|p|^2} = \infty$$

   The sum is over Gaussian primes with $\mathrm{Re}(p) > 0$ and $\mathrm{Im}(p) \geq 0$.

   (c) Show that there are infinitely many ordinary primes that are not Gaussian primes.

7. For $x \in \{0, 1, \ldots, n-1\}$ define $f(x) = r^x$. Suppose $r > 1$ is a real number. Show that $f(x)$ may be written as a sum of discrete Fourier modes (give a formula for the representation and the discrete Fourier modes) and find formulas for the coefficients.

Possible solution: The discrete Fourier modes are the functions $w_j(x) = e^{2\pi i j x/n}$. The discrete Fourier representation of f (if there is one) is

$$f(x) = \sum_0^{n-1} c_j w_j(x) \ .$$

This is supposed to be true for all $x \in \{0, 1, \ldots, n-1\}$. The discrete Fourier transform is the formula for c_j in terms of f:

$$\begin{aligned} c_j &= \frac{1}{n} \langle w_j, f \rangle \\ &= \frac{1}{n} \sum_{x=0}^{n-1} \overline{w_j(x)} f(x) \\ &= \frac{1}{n} \sum_{x=0}^{n-1} e^{-2\pi i j x/n} r^x \ . \end{aligned}$$

The last sum is a geometric series. If we define $z = e^{2\pi ij/n}r$, then the sum is

$$\sum_{x=0}^{n-1} z^x = \frac{1-z^n}{1-z} .$$

We know $z \neq 1$ because $|z| = r > 1$, by assumption. The discrete Fourier coefficients for this function are

$$c_j = \frac{1}{n}\frac{1-z^n}{1-z} = \frac{1}{n}\frac{1-e^{-2\pi ijn/n}r^n}{1-e^{-2\pi ij/n}r} = \frac{1}{n}\frac{1-r^n}{1-e^{-2\pi ij/n}r} .$$

8. Complete the table:

n	$\Lambda(n)$	$\pi(n)$	$\psi(n)$	$\phi(n)$
1	0	0	0	0
2	$\log(2)$	1	$\log(2)$	1
3	$\log(3)$	2	$\log(2)+\log(3)$	2
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Possible solution:

n	$\Lambda(n)$	$\pi(n)$	$\psi(n)$	$\phi(n)$
1	0	0	0	0
2	$\log(2)$	1	$\log(2)$	1
3	$\log(3)$	2	$\log(2)+\log(3)$	2
4	$\log(2)$	2	$2\log(2)+\log(3)$	2
5	$\log(5)$	3	$2\log(2)+\log(3)+\log(5)$	4
6	0	3	$2\log(2)+\log(3)+\log(5)$	2
7	$\log(7)$	4	$2\log(2)+\log(3)+\log(5)+\log(7)$	6
8	$\log(2)$	4	$2\log(2)+\log(3)+\log(5)+\log(7)$	4
9	$\log(3)$	4	$2\log(2)+2\log(3)+\log(5)+\log(7)$	6
10	0	4	$2\log(2)+2\log(3)+\log(5)+\log(7)$	4
11	$\log(11)$	5	$2\log(2)+2\log(3)+\log(5)+\log(7)+\log(11)$	10
12	0	5	$2\log(2)+2\log(3)+\log(5)+\log(7)+\log(11)$	4
13	$\log(13)$	6	$2\log(2)+2\log(3)+\log(5)+\log(7)+\log(11)+\log(13)$	12

9. Show that if $(n,a) > 1$, then there are not infinitely many primes in the arithmetic progression $a, a+n, a+2n, a+3n, \ldots$.

Possible solution: If $(n, a) > 1$ then there is a p that divides both a and n. This p therefore divides $a + kn$ for every k. Therefore, the only possible prime in the arithmetic progression is p.