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Section 6, The zeta function and complex analysis

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1 Introduction.

The Riemann zeta function is one of the most interesting objects in mathematics. Much of complex analysis was developed to study it. This section continues the discussion of complex analysis and the things you can learn about the zeta function.

2 Complex analysis facts

2.1 The Liouville theorem

A complex function $f(z)$ is *entire* if it is defined for all $z \in \mathbb{C}$ and has no poles. Actually, being defined sort of means having no poles. The basic *Liouville theorem* is that if f is bounded than f is constant. We suppose there is an M with $|f(z)| \leq M$ for all z and we conclude that $f'(z) = 0$ for all z. The proof is an application of the Cauchy formula for the derivative

$$f'(z) = \frac{1}{2\pi i} \oint_{|w-z|=r} \frac{f(w)}{(w-z)^2}\, dw \; . \tag{1}$$

We saw that you can apply absolute values in complex integrals as you can for real integrals. Or you can parametrize the contour integral, as in $w(t) = z + re^{2\pi i t}$ for $0 \leq t \leq 1$, and $dw = 2\pi i r e^{2\pi i t} dt$. Either way, you get

$$\begin{aligned} |f'(z)| &\leq \frac{1}{2\pi} \int_{|w-z|=r} \frac{|f(w)|}{r^2} \; |dw| \\ &\leq \frac{1}{2\pi} \frac{M}{r^2} 2\pi r = \frac{M}{r} \; . \end{aligned}$$

We see that $f' = 0$ by taking $r \to \infty$. It's good that we have to take $r \to \infty$, because bounded functions don't have to be constant unless they're defined in the whole complex plane. The conclusion is that bounded entire functions are all trivial – there are no interesting examples.

There is a generalization that applies to entire functions with "polynomial growth" at infinity. Polynomial growth means that there is an M and a p with

$$|f(z)| \leq M(1 + |z|)^p \; . \tag{2}$$

The Liouville theorem is that the only functions like that are actual polynomials. The Cauchy formula for that conclusion is (differentiate (1) with respect to z)

$$\left|f^{(n-1)}(z)\right| = \frac{(n-1)!}{2\pi i} \oint_{|w-z|=r} \frac{f(w)}{(w-z)^n}\, dw \; .$$

The original Liouville theorem was based on (1), which has $n = 2$. Suppose, for example, f has a polynomial bound with $p = 1$. Then we take $n = 3$ and learn that

$$|f''(z)| \leq \frac{2}{2\pi} \frac{M(1+|z|+r)}{r^3} 2\pi r = \frac{2M(1+|z|+r)}{r^2} \; .$$

For any z, the right side goes to zero in the limit $r \to \infty$. If $f''(z) = 0$ for all z, then $f(z) = az + b$. The conclusion is that if f "looks like" a polynomial of degree p in that it satisfies (2), then f is actually a polynomial of degree p.

Here's a curious corollary that we will need. If we want to show that $f(z)$ is a linear function of z, we don't have to prove (2) with $p = 1$. Any $p < 2$ will do. Suppose we have "polynomial" growth with power $p = 1.5$ (say). Then we also have polynomial growth with power $p = 2$ (because (2) with $p = 1.5$ implies (2) with $p = 2$), and therefore the fact that $f(z)$ is (at most) a quadratic polynomial in z. But if $|f(z)| \leq M(1 + |z|)^{1.5}$, then the quadratic term must vanish. This means that $f(z)$ is actually a linear "polynomial". It also implies that $|f(z)| \leq M'(1+|z|)$. That is, there exists an M', but we don't know much about how M' is constrained by M, if at all.

There are "real" versions of Liouville theorems that apply to harmonic functions. A real function $u(x, y)$ is *harmonic* if

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; .$$

If $f(z)$ is analytic and $u(x, y) = \mathrm{Re}(f(x+iy))$, then the Cauchy Riemann equations imply that u is harmonic (as we saw – look it up if you don't remember). The Liouville theorem for harmonic functions is the same as for analytic functions. If u is bounded then u is constant. If u has polynomial growth of degree p (i.e., $|u(x, y)| \leq M(1 + x^2 + y^2)^{p/2}$) then u is a polynomial of degree p. The proofs are similar to the ones we just saw. They are based on the *Poisson kernel* representation of harmonic functions. The following formula is copied from a complex analysis textbook (the one by Levinson and Redheffer, where it's formula (4.11) of chapter 6). It represents $u(x, y)$ on a circle of radius r in terms of the values of u on a circle of radius $R > r$:

$$u(r\cos(\theta), r\sin(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2}\, u(R\cos(\phi), R\sin(\phi))\, d\phi \; .$$

It's in a complex analysis book because it's derived from the Cauchy integral representation. The *Poisson kernel* is the part of the integrand that doesn't depend on u:

$$K(r, \theta, R, \phi) = \frac{1}{2\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta-\phi) + r^2} \; .$$

You can differentiate this with respect to x and y to get formulas for the derivatives of u, and imitate the steps above to prove Liouville for harmonic functions. It should take you less than an hour to do this.

The Liouville theorem we need for the application to the zeta function is a combination of these theorems. Suppose we don't at first know that $f(z)$ has polynomial growth, but only that the real part does. This would be

$$|\mathrm{Re}(f(z))| \leq M(1 + x^2 + y^2)^{p/2} \ . \tag{3}$$

If $p < 2$, this bound implies that $u(x, y) = \mathrm{Re}(f(x + iy))$ is linear: $u(x, y) = ax + by + c$. From this, the Cauchy Riemann equations allow us to show that the imaginary part $v(x, y)$ also is linear. To start,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = a \ ,$$

so

$$v(x, y) = ay + d(x) \ .$$

We don't know much about $d(x)$, except that it is differentiable because f is differentiable. The other Cauchy Riemann equation leads to $v(x, y) = -bx + e(y)$. Comparing these gives $v(x, y) = -bx+ay+h$, which is linear. This implies that $f(x + iy)$ is linear in x and y. Liouville's theorem then implies that $f(z)$ is a complex linear function of z. (Otherwise it might be that f is a real linear function of x and y but not a complex linear function.)

We have to go further still, and replace the two sided bound (4) by the *one sided* bound

$$\mathrm{Re}(f(z)) \leq M(1 + x^2 + y^2)^{p/2} \ . \tag{4}$$

This constrains how far in the positive direction u can go, but says nothing about how far in the negative u can go. It turns out that even a one sided bound with $p < 2$ implies that f is a linear function of z. The proof is that if u is harmonic and satisfies a one sided bound, then it satisfies a two sided bound. A harmonic function defined on all of $\mathbb{R}^2$ cannot go to infinity without also going to minus infinity.

This can be proved using the Poisson Kernel representation. First, subtract a constant $u(0, 0)$ from $u(x, y)$ so we get a harmonic function $u(x, y) - u(0, 0)$ that vanishes at the origin. Without loss of generality, we assume $u(0, 0) = 0$. Now, note two properties of the Poisson Kernel. One is that $K = \frac{1}{2\pi}$ if $r = 0$. The value $u(0, 0)$ is the simple average of $u(x, y)$ over the circle $x^2 + y^2 = R^2$. The other property is that K doesn't very too much when $r = \frac{1}{2}R$. There are constants $0 < C_1 < C_2 < \infty$ so that for all θ and ϕ,

$$C_1 \leq K(\frac{1}{2}R, \theta, R.\phi) \leq C_2 \ .$$

From these facts, we show that there is a C_3 so that

$$\max_{x^2+y^2=R^2} u(x, y) = A \implies \min_{x^2+y^2=(\frac{1}{2}R)^2} u(x, y) \geq -C_3 A \ .$$

This implies that
$$\min_{\frac{1}{2}R} u(x,y) \geq -C_3 M \left(1+R^2\right)^{p/2} \ .$$
It is an exercise to show that this implies that
$$|u(x,y)| \leq M' \left(1+x^2+y^2\right)^{p/2} \ .$$
This is the two-sided bound. The one sided bound implies the two sided bound.

We finally need to prove the lemma at the heart of the argument. For this, define $u_R(\theta) = u(R\cos(\theta), R\sin(\theta))$. The lemma is that if
$$\max_\theta u_R(\theta) = A$$
and
$$\int_0^{2\pi} u_R(\theta) d\theta = 0$$
then, for all θ,
$$u_{\frac{1}{2}R}(\theta) \geq -C_3 A \ .$$
To prove this, you can ask: what should u_R be to have mean zero and bounded from above by A to make the minimum over $\frac{1}{2}R$ as small as possible. The integral in question is
$$u_{\frac{1}{2}R}(\theta) = \int K(\theta, \tfrac{1}{2}R, \phi, R) u_R(\phi)\, d\phi \ .$$
There is an upper bound on u_R but no lower bound (not yet). After thinking about this for about a day, you realize that the "worst" thing to do is to make $u_R = A$ everywhere except at the ϕ value, ϕ_*, where $K(\theta, \frac{1}{2}R, \phi, R) = C_2 = \min$. You put a negative delta function mass there to achieve mean zero
$$u_R(\phi) = A - A\delta(\phi - \phi_*) \ .$$
Of course, the delta function isn't a real function, but you can use approximations to it.

Now for the point of all this. Suppose that $f(z)$ is an entire function with
$$|f(z)| \leq C_1 e^{C_2 |z|^p} \ . \tag{5}$$
Suppose $f(z) \neq 0$ for all z. Then $g(z) = \log(f(z))$ is well defined, and
$$\mathrm{Re}(g(z)) \leq \log(C_1) + C_2 \left|z\right|^p \ .$$
This is a one-sided bound for the harmonic function $u(x,y) = g(x+iy)$. The one-sided bound implies a two-sided bound with the same p but a possibly worse constant M. The two sided bound on u implies that u is a polynomial. This implies that g is a polynomial (look back for this). If $p < 2$, then g is a linear polynomial. This proves the theorem that was the purpose of this section: An entire function with exponential order p (this is the inequality (5) and $p < 2$ must be of the form
$$f(z) = e^{az+b} \ .$$
The only way for an entire function like $\xi(s)$ to be complicated is to have lots of zeros.

2.2 The argument principle

3 Exercises

1. Consider the function
$$g(z,s) = \prod_p \frac{1}{1 - zp^{-s}} \; .$$
   (a) Show that if $\mathrm{Re}(s) > 1$, then g is a meromorphic function of z and identify its poles.

   (b) g has a Taylor series expansion of the form
$$g(z,s) = 1 + G_1(s)z + G_2(s)z^2 + \cdots \; .$$
   Show that
$$G_1(s) = \sum_p p^{-s} \; ,$$
   and find expression for $G_2(s)$.

2. Consider the function
$$h(z,s) = \prod_p \left[\left(1 - zp^{-s}\right) e^{zp^{-s}} \right]$$
   (a) Show that this is an entire function of z if $\mathrm{Re}(s) > \frac{1}{2}$

   (b) Identify the coefficients in the Taylor series
$$h(z,s) = \sum_2^\infty c_n(s) z^n \; .$$

3. Suppose $f(\theta)$ is defined and continuous for $0 \leq \theta \leq 2\pi$ and $f(\theta) \leq A$ for all θ. Suppose that
$$\int_0^{2\pi} f(\theta) d\theta = 0 \; .$$
   Suppose that $r = R/2$ and define
$$g(\theta) = \int_0^{2\pi} K(r, \theta, R, \phi) f(\phi) d\phi \; .$$
   (a) Show that g is independent of R (if $r = R/2$).

   (b) Find a $C > 0$ independent of f and A so that $g(\theta) > -CA$. *Hint*: It suffices to take $\theta = 0$ and $A = 1$, why? Find the optimal C and find a sequence f_n that satisfy the hypotheses so that $g_n(0) \to -CA$ as $n \to \infty$.

(c) Finish the botched proof in the notes by showing that if u is harmonic and if $u(x,y) \leq C(1+x^2+y^2)^{p/2}$ then there is a C' so that $|u(x,y)| \leq C'(1+x^2+y^2)^{p/2}$.

4. Suppose ω_1 and ω_2 are non-zero complex numbers that are not co-linear in the complex plane (i.e., ω_2/ω_1 is not real). The *Weierstrass P function* is written $\mathcal{P}(z)$ and is defined by

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{(n_1,n_2)\neq(0,0)} \left(\frac{1}{(z - n_1\omega_1 - n_2\omega_2)^2} - \frac{1}{(n_1\omega_1 + n_2\omega_2)^2} \right) .$$

The sum (including the $1/z^2$ term) is a sum over *lattice points* in the complex plane. The description of $\mathcal{P}(z)$ for z near zero involves the lattice sums

$$G_r = \sum_{(n_1,n_2)\neq(0,0)} \frac{1}{(n_1\omega_1 + n_2\omega_2)^r} .$$

In your answers, be sure to distinguish the Greek letter ω from the Latin letter w.

(a) Show that the sum without *counter-terms* $1/(n_1\omega_1 - n_2\omega_2)^2$ does not converge abslutely, but converges absolutely if z is not a lattice point. *Hint*, compare

$$\frac{1}{(n_1\omega_1 + n_2\omega_2)^2} \qquad \text{to} \qquad \int_{x=n_1}^{n_1+1} \int_{y=n_2}^{n_1+2} \frac{1}{(x\omega_1 + y\omega_2)^2}\, dxdy$$

(b) Show that $\mathcal{P}(z+\omega_1) = \mathcal{P}(z)$, and similarly for ω_2. One way to do this is to show that $f(z) = \mathcal{P}(z+\omega_1) - \mathcal{P}(z)$ is bounded (has no poles, does not go to infinity as z goes to infinity).

(c) Show that $\mathcal{P}(z)$ is an even function of z and show that

$$\mathcal{P}(z) = \frac{1}{z^2} + Az^2 + Bz^4 + Cz^6 + \cdots .$$

Calculate A and B in terms of lattice sums G_4 and G_6. There are some big integers involved: 60 and 140.

(d) Suppose f is a meromorphic function with $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all z (except the poles). A *fundamental cell* for f is a parallelogram with corners a, $a+\omega_1$, $a+\omega_2$, $a+\omega_1+\omega_2$. Suppose that f has no poles on the boundary of the fundamental parallelogram. Show that that it is impossible for f to have a single simple pole in the interior of the fundamental parallelogram. *Hint*: Compute $\int_\gamma f(z)dz$ around the boundary of a fundamental parallelogram and see that the parts over opposite sides are related by periodicity of f.

(e) Show that $\mathcal{P}$ has one double pole in a fundamental parallelogram.

(f) Find D, and E so that

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - D\mathcal{P}(z) - E \; . \tag{6}$$

Hint: With the right value of D, the function $\mathcal{P}'(z)^2 - \left[4\mathcal{P}(z)^3 - D\mathcal{P}(z)\right]$ has no pole at $z = 0$ and therefore has no pole anywhere and therefore is a bounded entire function. You can figure out E by evaluating the difference at $z = 0$. Of course, D and E are also given in terms of lattice sums G_4 and G_6. The numbers can be big integers.

(g) Use the differential equation (6) to find a formula for C in terms of A and B and D and E. This gives a formula for G_8 (and all higher lattice sums, but that's not the assignment) in terms of G_4 and G_6. It may seem surprising that there is a formula for G_8 in terms of G_4 and G_6, but one explanation is that the lattice sums are determined by the two parameters ω_1 and ω_2. It's natural that two of the lattice sums determine ω_1 and ω_2, and therefore the rest of the lattice sums.