Quiz, Friday, Feb. 24, 12:30

1. Let x, y, and z be positive integers with no pair being relatively prime:

$$(x, y) > 1 \;, \quad (y, z) > 1 \;, \quad (x, z) > 1 \;.$$

   Prove either prove that there is an $n > 1$ that divides x, y, and z, or give a counter-example (positive integers x, y, and z with no $n > 1$).

2. Define the series a_n by

$$a_n = e^{\frac{1}{n}} - 1 - \frac{1}{n} \;.$$

   Show that the sum

$$\sum_1^\infty a_n$$

   converges absolutely.

3. Suppose a_n is defined for $n \geq 1$ and the partial sums are

$$S(x) = \sum_{n \leq x} a_n \;.$$

   (a) Assume that $|a_n| = O(n^r)$ with $r > -1$. Show that

$$|S(x)| = O(x^{r+1}) \;. \tag{1}$$

   (b) Give a counterexample to (1) if $r = -2$. That is, find a sequence a_n with $|a_n| = O(n^{-2})$ so that S_x is not $O(x^{-1})$. *Hint*: Is it necessary that $S_x \to 0$ as $x \to \infty$?

   (c) (*harder, do as time permits*) Prove the inequality (1) assuming that $r < -1$ and $S(x) \to 0$ as $x \to \infty$.

4. Define $t(n)$ to be the number of prime factors of n, counting repeats. This means that if $n = p_1^{r_1} p_2^{r_2} \cdots$, then $t(n) = r_1 + r_2 + \cdots$. For example, $t(72) = 5$ because $72 = 2^3 3^2$. Consider the product

$$f(s,x) = \prod_p \left(1 - xp^{-s}\right)^{-1}$$

   (a) Suppose that $s > 1$ and find the range of x values for which this product converges absolutely.

   (b) For s and x in this range, find a formula for $f(s,x)$ of the form

$$f(s,x) = \sum_1^\infty a_n(x) n^{-s} \; .$$

   Prove this formula as time permits. *Hint*: the formula for $a_n(x)$ involves $t(n)$.

5. List the elements of the multiplicative group G_{24} and determine $\phi(24)$. Do not determine the cycle structure of G_{24}.

6. Let $\chi(x)$ be any of the Dirichlet characters mod n. The discrete Fourier transform of χ is

$$w_j = \sum_{x=0}^{n-1} e^{-2\pi i j x/n} \chi(x) \; .$$

   Find a formula for

$$A = \sum_0^{n-1} |w_j|^2 \; .$$

   A is the same for every χ. The answer involves one of our number theoretic functions.