Quiz 2, Friday, March 24, 12:30

1. Consider the function $f(z) = ze^{\frac{1}{2}z^2}$. Note that $f(z) = \frac{d}{dz}e^{\frac{1}{2}z^2}$.
   (a) Suppose Γ is a contour parameterized by $\zeta(t) = it$ for $0 \leq t \leq 1$. Evaluate
   $$\int_\Gamma f(z)\,dz\;.$$
   (b) Is there a different (rectifiable) contour, Γ' from 0 to i for which
   $$\int_{\Gamma'} f(z)\,dz \neq \int_\Gamma f(z)\,dz\;.$$
   (c) Suppose
   $$g(z) = \frac{1}{z-1}e^{iz}\;,$$
   and that Γ is the same contour from 0 to i. Draw another (rectifiable) contour from 0 to i, Γ', that does not include 1, but so that
   $$\int_{\Gamma'} g(z)\,dz \neq \int_\Gamma g(z)\,dz\;.$$
   Explain your answer. Do not evaluate the integrals.

2. Show that
   $$\max_{|z|\leq 1} |\cos(z)| \geq \frac{3}{2}\;.$$
   Hint: Guess where the max is taken (it's not unique). Look at the signs of the Taylor coefficients of $(e^y + e^{-y})/2$ at $y = 0$.

3. The Jacobi theta function is
   $$\theta(s) = \sum_{-\infty}^{\infty} e^{-\pi n^2 s}\;.$$
   Show that $\theta(s)$ is an analytic function of s for $\sigma = \text{Re}(s) > 0$. Include as much detail as you have time for. If you use a theorem, state the theorem and all its hypotheses completely.

See problem 4 on the back

4. Consider the integral

$$u(r,\sigma) = \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{e^{r(s+s^{-1})}}{s^2}\, ds \; .$$

(a) Show that the integral converges absolutely for any real r if $\sigma > 0$.

(b) Show that the value of the integral is the same for every $\sigma > 0$. Give this argument in as much detail as time permits. Draw relevant contours, if necessary.

(c) Show that

$$\min_{\sigma > 0} \left[\max_{s = \sigma + it} \mathrm{Re}\left(s + s^{-1} \right) \right] = 2 \; .$$

(d) (Harder, do as time permits) Show that there is a C so that for every $r > 1$,

$$|u(r)| \leq C e^{2r} \; .$$