Numerical Methods II, Courant Institute, Spring 2020

http://www.math.nyu.edu/faculty/goodman/teaching/HonorsAlgebraII2020/NumericalMethodsII2020.html

Always check the `classes` message board before doing any work on the assignment.

Assignment 3, due April 14

Corrections: [none yet]

1. (*Normal* matrices) In case you forgot, a complex matrix H is *hermitian* if it is equal to its complex conjugate. In terms of entries, this is

$$H_{jk} = \overline{H}_{kj} \ .$$

More reminders: the eigenvalues of a hermitian matrix are real. A hermitian matrix has orthogonal eigenvectors that may be made orthonormal by normalizing. Let A be any $d \times d$ real matrix and let

$$H = \frac{1}{2}\left(A + A^t\right) \quad , \quad S = \frac{1}{2}\left(A - A^t\right) \ .$$

The matrix A is *normal* if it commutes with its transpose: $AA^t = A^t A$.

(a) H is the symmetric (hermitian) part and S is the skew symmetric part of A. Show that H is symmetric, S is skew symmetric, and iS is hermitian. [The i in the last formula is the reason for talking about complex hermitian rather than just real symmetric matrices.]

(b) Show that A is normal if and only if H and S commute.

(c) Suppose that H and KS are hermitian, and $Hu = \lambda u$ with $u \neq 0$. Show that if H and K commute then u is also an eigenvector of K.

(d) Assume that the eigenvalues of H are distinct. Show that A is diagonalizable and that the right eigenvector matrix R may be taken to be unitary.

(e) The *spectral radius* of A is

$$\rho = \max |\lambda| \ .$$

[The *spectrum* of a matrix is the set of eigenvalues. These are points in the complex plane. ρ is the radius of the smallest disk centered at the origin that contains the spectrum.] Show that if A is a normal matrix, then

$$\|A^n\|_2 = \rho^n \ .$$

The matrix norm on the left is the $2-$norm, which is defined by

$$\|A\|_2^2 = \max_{x \neq 0} \frac{(Ax)^t(Ax)}{x^t x} \ .$$

In terms of singular values, $\|A\|_2 = \sigma_{\max}$. *Conclusion*: Eigenvalue analysis correctly predicts stability and growth rates, even for large matrices, for normal matrices.

2. Let A be a $d \times d$ matrix with λ on the diagonal and 1 on the super diagonal, and zeros elsewhere. This is a $d \times d$ Jordan block with eigenvalue λ. Write this matrix as $A = \lambda I + J$, where J. The point of this problem is to show that the entries of A^n are much larger than λ^n. Let $m(n)$ be the largest element in A^n, that is: $m(n) = \max_{j,k} \left| (A^n)_{jk} \right|$. It is possible that $|\lambda|^n < 1$ and yet $m(n) \to \infty$ exponentially. We will find an approximate ("very approximate", which means not very approximate) formula for $m(n)$ that shows this. The tool is a simple form *Stirling's approximation*

$$m! \approx m^m e^{-m} \ .$$

Here is a sequence of steps/hints toward a solution:

- Figure out J^m and $(\lambda I)^k J^{n-k}$, assuming that $n < d$.
- Show that the binomial expansion

$$(\lambda I + J)^n = \sum_k \binom{n}{k} \lambda^k J^{n-k}$$

- Show that the binomial expansion gives an explicit formula for element of A^n.
- To find the largest element in A^n, maximize $\binom{n}{k}\lambda^k$. You can do this (approximately) by writing a formula for $\binom{n}{k+1}$ in terms of $\binom{n}{k}$ and using that to find k that satisfies

$$\binom{n}{k+1}\lambda^{k+1} = \binom{n}{k} |\lambda|^k \ .$$

  This is like calculus I, where you find a maximum by finding where the function is neither increasing nor decreasing. The k you get is probably not an integer, which you can worry about later.
- Use Stirling's approximation to get an approximate formula for $\binom{n}{k} |\lambda|^k$. It should involve $\left(\frac{n}{k}\right)^k$ and $\left(\frac{n}{n-k}\right)^{n-k}$.
- Insert your optimal k and simplify. The approximation $k \approx k+1$ should apply if k is large). The optimal k gives $n-k$ in terms of λ and k, which simplifies the expression.
- For fixed λ and large n (mathematicians say $n \to \infty$) with $n < d$ (that means large d too), the whole thing simplifies to something involving $|\lambda| + 1$ instead of $|\lambda|$.

3. A matrix is *highly non-normal* (unfortunate but unavoidable terminology) if $\|A^n\| \gg \rho^n$. Exercise 2 explored a special family of matrices that can be highly non-normal. These matrices are easy to analyze, but it might seem that the high non-normality is because of the Jordan structure. Use Python to explore matrices like in exercise 2, but with a on the sub

diagonal. Take $0 < \lambda < 1$ and a not large. Calculate ρ (let Python calculate ρ), and $\|A^n\|$ (whatever matrix norm you like). Demonstrate, computationally, that these matrices can have distinct eigenvalues and still be highly non-normal. *Warning*: Python will be unable to calculate the eigenvalues of A accurately unless d is rather small.

4. Find a second order accurate 3 point upwind method for the scalar linear advection equation $\partial_t u + s\partial_s u = 0$. The scheme has the form

$$u_{j,n+1} = au_{j-2,n} + bu_{j-1,n} + cu_{j,n} \ .$$

Assume the usual: $u_{j,n}$ is the approximation for $u(x_j, t_n)$ with $x_j = j\Delta x$ and $t_n = n\Delta t$. Assume $s > 0$ and $\lambda = s\frac{\Delta t}{\Delta x}$ is fixed as $\Delta x \to 0$. There are several ways to do this. One is to do quadratic interpolation in space at time t_n to get to the characteristic point $x_j - s\Delta t$. Another way is to do calculations like the Lax Wendroff calculations. Calculate the stability limit. What is the largest λ for which the scheme is linearly stable by von Neumann analysis?

5. The *shallow water equations* for two dimensional waves in three dimensional shallow water involve variables $h(x, y, t)$ (the height of the water surface over the bottom), $v_x(x, y, t)$ (the mean x velocity, depth averaged), and $v_y(x, y, t)$ (the y velocity, depth averaged). *Depth averaged* means the average of the three dimensional velocity V over the "water column":

$$v_x(x, y, t) = \frac{1}{\text{depth}} \int_{z=\text{bottom}}^{z=h(x,y,t)} V_x(x, y, z, t)\, dz \ .$$

The water is "shallow" when the x, y wavelength is large compared to the depth. The more "shallow" the water, the more accurate the equations. Tsunamis in the deep ocean are an example. The wavelength is several tens of kilometers and the depth is a few kilometers. The equations given below assume that the fluid density is $\rho = 1$, which is nearly true for sea water in CGS units. g is the gravitational acceleration constant, which is about 9.8 m/sec^2.

$$\begin{aligned}
&\partial_t h + \partial_x (hv_x) + \partial_y (hv_y) = 0 \\
&\partial_t (hv_x) + \partial_x \left(hv_x^2 + \frac{1}{2} gh^2 \right) + \partial_y (hv_x v_y) = 0 \\
&\partial_t (hv_y) + \partial_x (hv_x v_y) + \partial_y \left(hv_y^2 + \frac{1}{2} gh^2 \right) = 0
\end{aligned}$$

The *primitive variables* are the depth and velocity: h, v_x, and v_y. The *conserved variables* are depth (mass, because water is not compressed in this approximation) and momentum variables $m_x = hv_x$ and $m_y = hv_y$. These equations resemble the simple compressible gas equations given in class, but with "pressure" given by $\frac{1}{2}h^2$, and h taking the place of density.

The $h \sim$ density is supported by the idea that the amount of water above a two dimensional region of ocean, A, is (with unit density, as before)

$$\int_A h(x,y,t)\,dxdy\ .$$

In conserved variables, the equations are

$$\begin{aligned}
&\partial_t h + \partial_x\left(m_x\right) + \partial_y\left(m_y\right) = 0 \\
&\partial_t m_x + \partial_x\left(h^{-1}m_x^2 + \frac{1}{2}gh^2\right) + \partial_y\left(h^{-1}m_x m_y\right) = 0 \\
&\partial_t m_y + \partial_x\left(h^{-1}m_x m_y\right) + \partial_y\left(h^{-1}m_y^2 + \frac{1}{2}gh^2\right) = 0
\end{aligned}$$

Bathymetry refers to measurements of the depth of the ocean. The equations above are for ocean with a flat bottom, "constant bathymetry". Suppose $b(x,y)$ is the height of the bottom (probably negative, measured from where the surface would be if the water were flat. The "variable depth" shallow water equations, in conserved variables, are

$$\begin{aligned}
&\partial_t h + \partial_x\left(m_x\right) + \partial_y\left(m_y\right) = 0 \\
&\partial_t m_x + \partial_x\left(h^{-1}m_x^2 + \frac{1}{2}gh^2\right) + \partial_y\left(h^{-1}m_x m_y\right) = gh\partial_x b \\
&\partial_t m_y + \partial_x\left(h^{-1}m_x m_y\right) + \partial_y\left(h^{-1}m_y^2 + \frac{1}{2}gh^2\right) = gh\partial_y b\ .
\end{aligned}$$

The variable $h(x,y,t)$ still represents the height (depth) of the water above the bottom. This means that the water surface is flat if $b(x,y)+h(x,y,t) =$ const. You can check that flat water with no movement is a solution; if $h+b =$ const and $v_x = v_y = 0$ then all three equations are satisfied.

(a) Find the linearized equations assuming flat bathymetry and base constant solution $\overline{h} = h_0$, $\overline{v}_x = \overline{v}_y = 0$. Write them as a first order system with 3×3 matrices A_x and A_y. Show that the linearized equations have plane wave solutions that propagate in any direction at a wave speed $s_0 = \sqrt{gh_0}$. Do this by finding the eigenvalues and eigenvectors of the matrix $A_\omega = \omega_x A_x + \omega_y A_y$. Show that, from this analysis, the water moves in the same direction the wave is moving.

(b) Write a code to solve the linearized shallow water equations, assuming $h_0 = 3$ km. *Warning*: g changes when you express length in kilometers. Apply periodic boundary conditions in space with $h(x+L_x,y,t) = h(x,y+L_y,t)$ (and for the other variables). Use grid points $x_j = j\Delta x$, and $y_k = k\Delta x$ (same spacing in x and y). Choose $\Delta t = \lambda s_0 \Delta x$. Compute the space derivatives by second order or fourth order centered differences to get a semi-discrete scheme, then evolve in time using the four stage fourth order Runge Kutta method. Verify the code by computing plane waves in various directions (not just x or y or 45 degrees). Take an initial disturbance to be mode in direction ω with a Gaussian $e^{-u^2/2l^2}$ profile with a length $l = 30$ km and compute the propagation for a time that lets the waves move a distance of $3l = 90$ km. Do a convergence study to see that the scheme is second order accurate with second order space derivatives and fourth order with fourth order derivatives in space. Show that the scheme is stable for $\lambda < 2\sqrt{2}$ and unstable for $\lambda > 2\sqrt{2}$.

(c) Take bathymetry to represent an undersea island

$$b(x,y) = \frac{1}{2} h_0 e^{-r^2/2l^2} \; , \;\; r^2 = (x - x_0)^2 + y^2 \; .$$

Start with an initial plane wave moving in the $+x$ direction starting near $x = 0$ and choose $x_0 = 4l$. Make a movie of the solution,. For example, each frame can be a contour plot of $h(\cdot,\cdot,t)$. Choose L_x and L_y large enough to see the scattered wave well.

6. Write a code to apply the first order upwind scheme and the Lax Wendroff scheme to the scalar linear advection problem. Compute the modified equation for each scheme and show that the modified equation correctly predicts the qualitative behavior of the solution with piecewise constant initial data. You can solve the modified equation by a finite difference equation or using Fourier analysis.