PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: May 8, 2008.

Sample Final Exam

About the final exam

1. The exam will be closed book and closed notes. You will not need nor be allowed a calculator. You will be allowed one $8\frac{1}{2} \times 11$ piece of paper, a *cheat sheet*, with anything you like written on it. You may not use a magnifying glass to read it.

2. In the true/false section, first indicate your answer (true/false), then give a few words explaining the answer. A correct T/F without a reason will not get any points.

3. You may have points deducted for anything you write that is incorrect even if you also give the correct answer. Cross off anything you feel is incorrect.

4. You will receive 20% of the points for any question if you leave it blank or state that you are unable to answer it. You may have points subtracted for anything incorrect you write. If you have no idea how to answer a question, you will get more points by saying so than by giving a guess that is unlikely to be correct.

Sample questions

1. True/False (5 points each)

   (a) The forward equation for the probability density of a diffusion process always satisfies a maximum principle.

   (b) The Hamilton Jacobi Bellman equation is useless if it does not have a classical solution.

   (c) Fourier analysis is useful way to solve a linear PDE only if it has constant coefficients.

   (d) The final value problem for pricing American style options is nonlinear because it does not satisfy the superposition principle.

   (e) If $u(x,t)$ satisfies the heat equation, $\partial_t u = \frac{1}{2} \triangle u$, then $\|u(t)\|_{L^1} = \int |u(x,t)|\, dx$ does not change in time.

2. Show that the solution of the initial value problem $\partial_t u = \frac{1}{2}\triangle u$, with initial data $u(x,0) = f(x) = |x|$ in two dimensions has the form of a similarity solution $u(x,t) = t^p \phi(x \cdot t^{-1/2})$. Find the power, p. Find the PDE satisfied by ϕ. Show that the solution to this PDE is radially symmetric, i.e. a function of $|x| = r$. What extra conditions on ϕ do we need to specify in order to get the correct solution, u? Is there a simple closed form solution?

3. Formulate the PDE we have to solve to calculate

$$E\left[\exp\left(\int_0^\tau r(t)dt\right)S(\tau)\right] .$$

Where $\tau = \min(t)$ so that $r(t) = \mu$ and $dS = \mu S dt + \sigma S dW_1$ and $dr = a(\overline{r} - r)dt + \gamma dW_2$, where W_1 and W_2 are independent Brownian motions. Give the PDE, the initial or final data, the boundary conditions and the location of the boundary, if any. Assume $r(0) < \mu$. This is the present value of a contract to give you a share of stock at the first moment where the risk free rate is equal to the expected growth rate of the stock.

4. Find the ODEs needed to solve the initial value problem for $\partial_t u = (1 + x^2)\partial_x^2 u$ with $u(x, 0) = e^{\beta x}$. Explain how to use these solutions to solve the initial value problem with $u(x, 0) = x_+$ using the Fourier transform.

5. Suppose

$$\partial_t u = \frac{1}{2}\partial_x^2 u - \partial_x\left(x^3 u\right) .$$

   (a) If this is the forward equation for an SDE, what is the SDE?

   (b) Based on the direction of the drift, do you expect this SDE to have a steady state probability distribution as $t \to \infty$?

   (c) Find a function, $g(x) > 0$ so that if $u(x, 0) \le g(x)$ for all x, then $u(x, t) \le g(x)$ for all $t > 0$. Do this so that $g(x) \to 0$ as $x \to \pm\infty$ and $\int g(x)dx < \infty$. It is easiest if g is a symmetric function of x. Hint: this is a comparison principle. Design g so that when you plug it into the PDE (with $\partial_t g = 0$), you get a sign. How does this bound prevent probability from escaping to infinity?

6. Suppose x has n components and $f(x, t)$ satisfies (assuming the summation convention throughout)

$$\partial_t f + \frac{1}{2}D_{jk}\partial_{x_j}\partial_{x_k} f + a_j\partial_{x_j} f = 0 ,$$

for all x with $x_n > 0$ and has boundary conditions

$$v_j\partial_{x_j} f = 0 ,$$

when $x_n = 0$. This is called an *oblique derivative* boundary condition. Find the PDE and boundary conditions that $u(x, t)$ needs to satisfy in order that

$$\int u(x, t)f(x, t)\, dx$$

should be a constant for every such f.

7. Let $f(s,t)$ be the value of an Azorian call (terminology due to Goodman). An Azorian call is a standard European call except that the holder is required to exercise the option the instant $S(t)$ exceeds an agreed upon early exercise price $L > K$ (K being the strike price). Give the Black-Scholes PDE governing f including relevant final and boundary conditions.

8. This is a stochastic interest rate extension of one of the Merton optimal investment problems. Suppose there is a stock with price $S(t)$ and a short term interest rate $r(t)$ that satisfy

$$\begin{aligned} dS(T) &= \mu S(t)dt \ + \ \sigma S(t)dW_1(t) \ , \\ dr(t) &= a\left(\overline{r} - r(t)\right)dt \ + \ \gamma dW_2(t) \ . \end{aligned}$$

Here, $W_1(t)$ and $W_t(t)$ are correlated Brownian motions with correlation ρ. At time t, our wealth is $X(t)$. The amount in the stock is α and the amount in "overnight" debt is $(1-\alpha(t))X(t)$. This gives

$$dX \ = \ r(t)X(t)dt + (\mu - r(t))X(t)dt + \alpha(t)X(t)\sigma dW_1 \ .$$

We want to choose $\alpha(t)$ so as to optimize $E\left[U(X(T))\right]$.

   (a) Define the value function, $f(x,r,t)$, relevant to this optimization problem. Give the formal definition as the solution to an optimization problem at time t.

   (b) Give the Hamilton Jacobi Bellman backward equation that f satisfies. Be careful to think about the possibility that $r(t)$ will exceed μ some of the time.

   (c) Suppose the utility function is $U(x) = x^\gamma$. Show that $f(x,r,t) = x^\gamma g(r,t)$, for some function g (This is a scaling ansatz.). Find the PDE that g satisfies. Why is this PDE preferable to the one f satisfies?

9. Describe the superjet and jubjet at each point in the plane of the function $f(x,y) = \max(x^2 + y^2 - 1, 0)$.