PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected February 3, 2008. Corrections: question 6(c): (1) should changed to (2).

Assignment 1, due February 4

1. Let Z be a standard normal random variable with probability density $g(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$. The *cumulative normal* is the function

$$N(x) = \Pr(Z \le x) = \int_{-\infty}^{x} g(z) dz \; .$$

   (a) Make a rough sketch of $N(x)$ that shows: the symmetry (the relation between $N(x)$ and $N(-x)$, the places where the slope is large and small, and the approach to the limiting values as $x \to \pm\infty$.

   (b) Let $u(x,t)$ be the solution of the heat equation, in one space dimension, with initial data $f(x) = 0$ if $x < 0$, and $f(x) = 1$ if $x \ge 0$. Use the Green's function representation formula to show that

$$u(x,t) = N(xt^{-1/2}) \; . \qquad (1)$$

   Work the integral ((7) in Section 1 of Goodman's notes, (2) in Section (2) of Kohn's notes) and show it has the form (1).

   (c) Use (a) and (b) to show that $u(x,t) \to f(x)$ as $t \downarrow 0$.

2. This exercise explains the method of similarity solutions.

   (a) Show that if $u(x,t)$ satisfies the heat equation, then $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also is a solution.

   (b) Assuming that the solution to the initial value problem is unique, show that the solution, u, from Problem 1. satisfies $u_\lambda(x,t) = u(x,t)$ for any $\lambda > 0$. Hint: what initial value problem does u_λ satisfy?

   (c) Show that if $u_\lambda(x,t) = u(x,t)$ for all $\lambda > 0$, then there is a function $v(y)$ so that $u(x,t) = v(xt^{-1/2})$.

   (d) Assume that $u(x,t) = v(xt^{-1/2})$ and u satisfies the heat equation. Find the ordinary differential equation that $v(y)$ satisfies.

   (e) Use the method of integrating factors from ordinary differential equations to show that $v'(y) = Ce^{-y^2/2}$, and therefore that $v(y) = a + bN(y)$ for some constants a and b.

   (f) Use the initial conditions and the behavior of the cumulative normal to determine a and b and recover the solution of Problem 1.

   (g) Explain the statement that the solution to this initial value problem at time t varies on a length scale $\lambda(t) = \sqrt{t}$, i.e. is rapidly varying for small time and slowly varying for large time.

3. Suppose $u(x,t)$ satisfies the heat equation. Calculate

$$\partial_t \int_{R^n} |x|^2 \, u(x,t) \, dx \; .$$

Use the result to explain the statement that diffusion makes particles wander away from the origin. As a point of information, not an action item, if u is a probability density for a Brownian motion particle, the same result may be derived from Ito's lemma.

4. Suppose $u(x,0) = f(x)$, u satisfies the heat equation, and $\int_{R^n} |f(x)| \, dx < \infty$. Show, using the Green's function representation of the solution, that $M(t) = \max_x |u(x,t)|$, satisfies $M(t) \leq Ct^{-n/2}$. Hint: if $I = \int f(x)g(x) \, dx$, then $|I| \leq \int |f(x)| \, dx \cdot \max_x |g(x)|$. There are three related takeaways here. First, it shows that $M(t) \to 0$ as $t \to \infty$. Diffusion spreads the material thinner as time goes on so that the maximum density goes to zero. Second, the density is bounded for any $t > 0$ even if f is not bounded. Any infinite spikes in f evolve into merely large spikes at any positive time, and eventually to very small ones at large time. Third, there are no finite mass steady states (which we also could have learned from Problem 3).

5. Suppose $v(x)$ is a vector field defined for all x, and the total flux is a sum of a diffusive flux and an *advective* flux:

$$F_{tot}(x,t) \;=\; F_d(x,t) \;+\; F_a(x,t) \;=\; -\frac{1}{2}\nabla u(x,t) \;+\; v(x)u(x,t) \; .$$

   (a) Neglect the diffusive flux and explain why the advective flux corresponds to molecules of sugar (or whatever is diffusing) at x being carried along with velocity $v(x)$.

   (b) Derive a PDE satisfied by u assuming this flux and conservation.

   (c) Let $v(x) = ax$, where a is a constant. For one sign of a, this carries particles away from the origin. For the other sign, it carries them toward the origin. If particles start near the origin, we expect diffusion, on average, to take particles away from the origin. A steady state a solution of the PDE from part b. that satisfies $\partial_t u = 0$, i.e. a function of x alone. Which sign of a would you expect to allow to support steady states?

6. We seek a steady state for the PDE in Problem 5 that is radially symmetric: $u(x) = g(|x|)$.

   (a) Show that ∇u points in the radial direction (directly toward or away from the origin) and $|\nabla u(x)| = |g'(r)|$, where $r = |x|$.

   (b) Show that the total flux vanishes, $F_{tot} = 0$, if and only if

   $$\frac{1}{2} g'(r) \pm arg(r) = 0 \ . \tag{2}$$

   Determine the sign.

   (c) Find the general solution of (2) and the corresponding steady state solutions $u(x)$, when the exist. Is your answer consistent with Problem 5c? Note that larger a (of the proper sign) causes the steady state to be more tightly concentrated around the origin, while smaller a allows more wandering. Is this consistent with intuition (not an action item).