PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: February 5, 2008. The definition of the hat function was fixed. In Problem 2, $I(a)$ became $I(c)$. In Problem 3, the big formula for $h(x)$ was fixed.

Assignment 2, due February 11

1. Calculate the Fourier transform of $f(x) = e^{-|x|}$. Now calculate the Fourier transform of $g(x) = 1/(1+x^2)$. (Hint: the direct and inverse Fourier transform formulas ((8) and (9) from Section 2 of Goodman's) are similar, but not exactly the same.) Explain how this calculation illustrates the relationship between the smoothness of a function and the decay of the transform. To start, note that f is rapidly decaying but not smooth.

2. Calculate
$$I(c) \;=\; \int_{-\infty}^{\infty} e^{cx} e^{-x^2/2}\, dx \; .$$
Assuming the result applies for complex values of c, calculate the Fourier transform of $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$.

3. The *convolution* of functions f and g is
$$h(x) \;=\; (f \star g)\,(x) \;=\; \int_{-\infty}^{\infty} f(x-y)g(y)\, dy \; .$$
Use the Dirac formula (12) from Section 2 to find a formula for $\widehat{h}$ in terms of $\widehat{f}$ and $\widehat{g}$. Hint: if $f(x) = \frac{1}{\sqrt{2\pi}} \int e^{ipx} \widehat{f}(p)dp$ and $g(x) = \frac{1}{\sqrt{2\pi}} \int e^{iqx} \widehat{g}(q)dq$, then you can simplify
$$h(x) \;=\; \frac{1}{2\pi} \iiint e^{ip(x-y)} e^{iqy} \widehat{f}(p)\widehat{g}(q)\, dydpdq \; .$$
Now use the uniqueness theorem.

4. Suppose $g(x) = f(x/L)$. Find the formula for $\widehat{g}$ in terms of $\widehat{f}$. Use this to compute the function whose Fourier transform is $e^{-p^2/2t}$, given that we know the answer when $t = 1$.

5. Use the 3 above results to derive the heat kernel representation of the solution of the heat equation ((7) from Section 1) from (15) of Section 2.

6. Adapt the method of images (Kohn's notes, Section 2) to find the solution of the following initial boundary value problem for the heat equation. $u(0,t) = 1$, for $t > 0$, $\partial_t u = \frac{1}{2}\partial_x^2 u$ for $x > 0$ and $t > 0$, and $u(x,0) = 0$ for $x > 0$. Use your solution formula for the following:

   (a) For which x value is $|\partial_x u|$ largest?

   (b) Sketch (on the same graph) the solution at an early time and a later time.

   (c) Show that $u(x,t) > 0$ for all $x > 0$ and $t > 0$. This is called *infinite propagation speed* of heat. Why?

   (d) Show that $u(1,t)$ is exponentially small for small t, which mitigates the disturbing answer to part c a little. You may use the approximation, which holds when z is large:

$$1 - N(z) \;\approx\; \frac{1}{z\sqrt{2\pi}} e^{-z^2/2} \;.$$

7. The *hat function* is $f(x) = (1 - |x|)_+$. Explore the accuracy of approximating the hat function using the FFT. Take the FFT of a periodic function that is equal to the $f(x)$ on the interval $[-R, R]$ using N equally spaced points. Use the coefficients to construct an approximation (why π instead of 2π?)

$$\widetilde{f}_{R,N}(x) \;=\; \sum_{\alpha} \widehat{f}_{R,N,\alpha} e^{i\pi\alpha x/R} \;.$$

   Make plots of f and $\widetilde{f}_{R,N}$ for various values of R and N to see what you have to do to get a good approximation. At the same time, make plots of the difference $\widetilde{f}_{R,N} - f$. Note where the difference is largest.

8. (not to hand in) If you are frustrated debugging your work for problem 7, try using the same code to get the known Fourier coefficients of a Gaussian $f(x) = e^{-x^2/2}$. This will help get the normalizations right (powers of N and 2π.