PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: February 20, 2008.

Assignment 3, due February 25

1. Suppose $V(t)$ is a control volume at each time t and that $\Gamma(t)$ is the boundary of $V(t)$. Let $n(x)$ be the outward unit normal to V at each point, x, or Γ. Suppose $r(x)$ is a scalar normal velocity field that represents the at which Γ is moving in the normal direction. That is, $y \in \Gamma(t)$ if and only if $y = x + tr(x)n(x)$ for some $x \in \Gamma(0)$.

   (a) Draw a picture of this situation in two dimensions for a small (but not too small) value of t. Let $r(x)$ be positive for some x and negative for other x so that $\Gamma(t)$ is partly within $V(0)$ and partly outside it.

   (b) Draw a picture that proves (or makes very plausible) the formula

   $$\frac{d}{dt} \int_{V(t)} f(x)\, dx \bigg|_{t=0} = \int_{\Gamma(0)} r(x) f(x)\, dA(x) \; . \tag{1}$$

   (c) Now suppose the points of $\Gamma(t)$ are advected by a velocity field $b(x)$. Use (1) to derive the formula

   $$\frac{d}{dt} \int_{V(t)} f(x)\, dx \;=\; \int_{\Gamma(t)} \left(b(x) \cdot n(x) \right) \, f(x)\, dA(x) \; . \tag{2}$$

   (d) Generalize (2) to

   $$\begin{aligned} \frac{d}{dt} \int_{V(t)} u(x,t)\, dx \quad &= \quad \int_{\Gamma(t)} \left(b(x) \cdot n(x) \right) \, u(x,t)\, dA(x) \\ & \qquad + \int_{V(t)} \partial_t u(x,t)\, dx \; . \end{aligned} \tag{3}$$

   (e) Use (1) and (2) of Section 3 of the notes, together with (3) here to show that if $\Gamma(t)$ and $u(x,t)$ are advected with the same velocity field $a(x)$, then

   $$\frac{d}{dt} \int_{V(t)} u(x,t)\, dx \;=\; 0 \; . \tag{4}$$

   (f) Comment on the relation between (4) here and conservation of a quantity advected by velocity field $a(x)$. This is a short answer question, very short.

2. In one dimension suppose that the advection velocity is $a(x) = \alpha x$. This is compressive if $\alpha < 0$ and expansive if $\alpha > 0$. We want to solve the PDE $\partial_t u + \partial_x(\alpha x u) = 0$ with initial data $u(x, 0) = f(x) = e^{-x^2/2}$.

   (a) Define the characteristic curves to be $x(y, t)$. These satisfy $\partial_t x(y, t) = a(x(y, t), t)$ and initial condition $x(y, 0) = 0$. In other words, $x(y, t)$ is the characteristic curve that starts at y at time $t = 0$. Find an explicit formula for $x(y, t)$. This is sometimes called the *flow map*. The inverse function, which is the $y(x, t)$ so that $x(y(x, t), t) = x$ is the *Lagrangian variable* (or *Lagrangian coordinate*).

   (b) Calculate the quantity $A(y, t) = \text{div}a(x(y, t))$. Calculate the corresponding $M(y, t)$ using (6) from Section 3.

   (c) Find the Lagrangian coordinate explicitly in this case and use it to find a formula for $u(x, t) = M(y(x, t), t)$.

   (d) Check that your answer to part (c) satisfies the advection equation and the initial conditions.

   (e) Give another check of your answer by finding a probabilistic interpretation. Since the flow map is linear, if we choose $X(0)$ to a standard normal (mean zero variance one), then $X(t)$ will be normal with mean zero and another variance. Find that variance and show that the corresponding probability density is the result of part (c).

3. The maximum principle and related comparison principles apply to more general equations of the type (10) in Section 3 of the notes. This exercise sketches a proof. We must use the hypotheses that $a(x)$ is symmetric and positive definite at each point. The proof is almost exactly the same as the one for the Laplace and heat equations, except some linear algebra is more complicated.

   (a) Show that if A and B are any $n \times n$ matrices, then $\text{tr}(AB) = \text{tr}(BA)$. Hint: both are equal to $\sum_{ij} A_{ij} B_{ij}$.

   (b) Show that if A and B are symmetric with and positive semi-definite (has no negative eigenvalues), then $\text{tr}(AB) \geq 0$ with strict inequality if A and B are positive definite. Hint: write $B = Q\Lambda Q^t$, where Q is orthogonal and Λ is diagonal with non-negative entries. Then $\text{tr}(AB) = \text{tr}(AQ\Lambda Q^t) = \text{tr}(Q^t AQ\Lambda)$ (using part (a) and the fact that matrix multiplication is associative).

   (c) Write (10) of Section 3 in the form $\mathcal{L}u = \text{tr}(a(x)D^2 u) + b(x) \cdot \nabla u + c(x)u = f$ and suppose $c(x) \leq 0$ for all x and $f(x) > 0$. Show that if $u(x_*) \geq$ is the maximum of u over $V \cup \Gamma$ and, then $x_* \in \Gamma$.

   (d) Prove that the boundary value problem for $\mathcal{L}u = f$ $x \in V$, $u(x) = g(x)$ for $x \in \Gamma$, has a unique solution among functions that are C^2 in V and continuous in $V \cup \Gamma$. Use the (somewhat necessary) hypothesis that $c(x) \leq 0$ for all $x \in V$. Hint: you need to prove the weak maximum principle for functions that satisfy $\mathcal{L}u = 0$ in V and $u = 0$

in Γ. You need to show that that maximum cannot be greater than zero, which is part (c).

(e) Prove the corresponding uniqueness theorem for solutions of the initial value problem $\partial_t u = \mathcal{L}u + f$. You can avoid the hypothesis $c \leq 0$ calculating the PDE satisfied by $v(x,t) = u(x,t)e^{-\lambda t}$.

(f) If this were a pure math PDE class, we could go on proving stuff like this all semester. Let's move on instead.