PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: February 27, 2008.

Assignment 4, due March 3

1. Take the Fourier transform of both sides of equation (6) in Section 4 of the notes. Let $\widehat{u}(p,t)$ be the Fourier transform of u in the x variable only. You should get a PDE for $\widehat{u}$ that involves first derivatives in p and t. Show that when you solve this equation by the method of characteristics you get the equations (7).

2. Look for a solution of (6) of the form

$$u(x,t) \;=\; A(t)e^{-c(t)(x-\mu(t))^2/2} \; .$$

   Write the equations for the evolution of the parameters $c(t)$, $\mu(t)$, and $A(t)$. Show that this is equivalent to the ansatz (11) and that the c equation is the same (which is the reason for keeping the notation). Find the explicit solution to the c, μ, A ODEs that has $c \to -\infty$ and $u(x,t) \to \delta(x-y)$ as $t \to 0$ (y is a parameter).

3. The Green's function (10) satisfies the PDE (6) in the variables x and t. We will see that it also satisfies a PDE in the y and t variables. This is the *adjoint* PDE. If $Lu = a_{ij}(x)\partial_{x_i}\partial_{x_j}u + \cdots$, then the *adjoint*, L^*, is defined by the relation

$$\langle u, Lv\rangle \;=\; \langle L^*u, v\rangle \; ,$$

   which is supposed to hold for all functions u and v that are C^2 (all partials of order 2 continuous functions of x) and that decay rapidly enough as $x \to \infty$ (together with partials up to order 2) so that there are no boundary terms when you integrate by parts. The L^2 *inner product* is the integral

$$\langle u, v\rangle \;=\; \int_{R^n} \overline{u}(x)v(x)\, dx \; .$$

   Here $\overline{u}$ is the complex conjugate of u if u is complex. Obviously, you don't need it if u and v are real. You don't need complex valued functions when computing the adjoint of a *differential operator*, like L, with real coefficients.

   (a) Give an abstract proof (i.e. not one related to second order differential operators) that shows that the adjoint of L^* is L.

   (b) Compute the adjoint of the operator (summation convention in use)

$$Lu \;=\; a_{ij}(x)\partial_{x_i}\partial_{x_j}u \;+\; b_i(x)\partial_i \;+\; c(c)u \; .$$

   (c) Identify a specific operator by writing (6) in the form $\partial_t u = Lu$. Compute the adjoint of this specific L.

(d) Show by direct calculation that the Green's function (10) satisfies $\partial_t G(y,x,t) = L_y^* G(y,x,t)$. Here, L_y^* means the adjoint of L but operating in the y variable. For example, if $Lu = x\partial_x u$, then $L_y^* G = -\partial_y \left(yG(y,x,t) \right)$.

(e) Show directly that as a function of y for each fixed x, $G(y,x,t) \to \delta(y-x)$ as $t \to 0$. Use this to write the solution of the initial value problem for L^* as an integral involving G.

4. A simple short rate interest rate model prices bonds and interest rate derivatives using the PDE

$$\partial_t u \;=\; \frac{r}{2}\partial_r^2 u \;+\; \lambda\left(\overline{r} - r\right)\partial_r u + ru \; .$$

This PDE has solutions of the form $u(r,t) = A(t)e^{B(t)r}$. Find the differential equations B and A must satisfy. Solve them. This is a simpler version of the Heston calculation in the notes.