PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: March 3, 2008.

Assignment 5, due March 10

1. The Ornstein Uhlenbeck process $dX = -\alpha X dt + dW$ has solution

$$X(t) \;=\; e^{-\alpha t} X_0 \;+\; \int_0^t e^{-\alpha(t-s)}\, dW(s)\;. \tag{1}$$

   (a) Show that the solution formula (1) is the solution of the Ornstein Uhlenbeck process.

   (b) For fixed X_0 not random, show that the probability distribution of $X(t)$ is Gaussian. Identify the mean and variance using (1) and the Ito isometry formula

$$E\left[\left(\int_0^t m(s)\, dW(s)\right)^2\right] \;=\; \int_0^t E\left[\, m(s)^2 \,\right]\, ds\;.$$

   (The $m(s)$ here is not random. We don't need the full power of the Ito isometry formula or the expectation on the right side.)

   (c) Let $u(x_0, x, t)$ be the probability density that is normal with that mean and variance. Show by comparing formulas that this is the Green's function for the PDE $\partial_t u - \alpha \partial_x (xu) = \frac{1}{2}\partial_x^2 u$. We found that Green's function using the exponential ansatz and Fourier transform last week. You just need to show that the u above is the same as that u.

2. There is a Merton model of corporate bond default that works as follows. The value of the firm at time is $V(t)$ and satisfies (what else?) $dV = \mu V dt + \sigma V dW$. The firm defaults if there is a $t \leq T$ with $V(t) < B$. (The *equity* of the firm is $V - B$. This is the value of the firm to its owners. If that value is negative, the owners don't want it.)

   (a) Formulate an initial boundary value problem for the survival probability density $p(v,t)$ given that the initial value of the firm is V_0.

   (b) Convert this a constant coefficient PDE with the usual substitution $x = \ln(v)$.

   (c) Remove the first derivative (advection) term with the substitution $p(x,t) = e^{\lambda x} q(x,t)$ using the appropriate value of λ. (This is the PDE version of a well known trick using Girsanov's formula, but this time, do it the PDE way.)

   (d) Use the method of images to write the solution to the initial boundary value problem for q and therefore for p.

(e) Express the survival probability in terms of the original parameters V_0, μ, σ, T, and B, and the cumulative normal $N(x) = P(Z \leq x)$, where $Z \sim \mathcal{N}(0,1)$.

(f) Let $F(v,t)$ be the survival probability starting with $V(t) = v$. Give a backward equation formulation – final values, boundary conditions, PDE – that determines F.

(g) Adapt the strategy in parts (b) – (e) to write a solution to this backward equation. Check that it agrees with the forward equation answer.

3. Many proposed dynamic trading strategies are of the *singular boundary control* type. These apply no control unless the portfolio is on the boundary of some allowed set. The simplest such strategy is a portfolio accumulation strategy under which the portfolio consists of a changing number of shares of a single stock that evolves according to geometric Brownian motion. If $N(t)$ is the number of shares (not restricted to integer values) then $Y(t) = N(t)S(t)$ is the "portfolio" value at time t. As usual, $dS = \mu S dt + \sigma S dW$. We suppose the trader does nothing if $Y(t) > B$ but buys stock whenever $Y(t) = B$ to insure that $Y(t) \geq B$ for all t.

(a) Write a forward PDE, with initial and boundary conditions, for $Y(t)$ that uses the fact that N is constant for $Y > B$ and the probability flux vanishes when $Y = B$.

(b) Let $F(y,t)$ be the expected value of $Y(T)$ given that $Y(t) = y$. Write a backward equation PDE formulation (final conditions and boundary conditions) that determines F. (Hint: for the boundary condition, start with the forward problem (a) and use the duality method from class and Kohn's notes, Section 1 page 14.)