PDE in Finance, Spring 2008,
http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html
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Assignment 6, due April 14

1. We explore the economic consequences of the solution (12) in Section 6. Note that increasing r makes saving (low consumption) pay off more. Higher ρ makes Alice less interested in the future and therefore less interested in saving.

   (a) Find a formula for b, as a function of T and the parameters, so that $B(T) = 0$.

   (b) Discuss the behavior of b and the solution (12) as $T \to \infty$ in case I: $\rho - \gamma r > 0$, and case II: $\rho - \gamma r < 0$. Give an economic (pseudo economic) explanation of the difference between the large horizon (T) solutions in the two cases.

   (c) Give qualitative sketches of the optimal $\alpha(t)$ and $X(t)$ with $X(0) = 1$ in the two cases for large T.

2. Consider the optimal control problem

$$\min \int_0^T \left(X(s)^2 \; + \; \left(\alpha(s)^2 - 1\right)^2 \right) ds \; ,$$

   with dynamics $dX = \alpha(t)dt$ and $X(0) = 0$. Show that the minimum value is zero, but that the minimum is not attained by any continuous function $\alpha(t)$. Give a sequence of approximate optimal controls so that the performance converges to zero.

3. Here is a deterministic continuous time version of the optimal execution model of Almgren and Chriss. Your job is to sell a large block of N shares of stock. The present price is $y_0 = Y(0)$. In our continuous time model, shares are sold at a rate of α shares per unit time. We want the money quickly, but selling too fast moves the market. We want a good tradeoff. Following Almgren and Chriss, we consider two kinds of market impact, temporary and permanent. The temporary impact with parameter a means that the income in time dt from selling at rate α is $dI = \alpha Y(1 - a\alpha)dt$. This models temporarily lowering the price from Y to $Y(1 - a\alpha)$ without effecting the price at later time. Permanent impact lowers the stock price for the long term, which we model with $dY = -b\alpha Y dt$. The number of shares remaining to be sold is $N(t)$, which satisfies $dN = -\alpha dt$. The discounted total income is

$$J \;=\; \int_0^\infty e^{-\rho s} dI(s) \;=\; \int_0^\infty e^{-\rho s} \alpha(s) Y(s)(1 - a\alpha(s))ds \; .$$

   The value function is $u(y, n) = \max J$, under the conditions that $N(0) = n$ and $Y(0) = y$.

Note: the temporary and permanent market impacts are proportional to the stock price (unlike the earlier version). Proportionate impact is marginally more realistic and makes the optimization problem simpler.

(a) Derive a Hamilton Jacobi equation of the form

$$0 \;=\; \max_{\alpha}\left\{\; (**)\alpha\partial_n u \;+\; (***)\alpha\partial_y u \;+\; y\alpha - a\alpha^2 \;-\; \rho u \;\right\} \qquad (1)$$

(b) Find the optimal α_* as a function of u and its derivatives. Using this α_*, rewrite the Hamilton Jacobi equation in the form

$$Cyu \;=\; (\, y - \partial_n u - ay\partial_y u\,)^2 \;. \qquad (2)$$

Find a formula for C in terms of the parameters ρ and b.

(c) Describe boundary conditions that determine u when $y = 0$ or $n = 0$.

(d) Make an argument that for any fixed n, u should be proportional to y.

(e) Substitute the corresponding ansatz, $u(y, n) = yv(n)$, into (2) and find an ordinary differential equation for v of the form

$$\partial_n v \;=\; C_0 \;+\; C_1\sqrt{v} \;+\; C_2 v \;, \qquad (3)$$

with formulas for the three constants. What is the initial condition for v? You do not need to find an explicit formula for v. That seems difficult or impossible.

(f) Make an intuitive (not very mathematical) argument for the correctness of the value of C_0 by arguing how much market impact there needs to be for small n. How/why does the optimal strategy small n avoid market impact and the effect of discounting?

(g) Determine the behavior of $v(n)$ as $n \to \infty$. What does this say about the value of a very large holding of a single traded asset (in this model!)?

(h) (FYI only, not an action item) The actual Almgren and Chriss model allows the stock price to vary randomly. This would change the answer to (g). It has several other differences that are more technical.