PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: April 5, 2008.

Assignment 7, due April 14

1. Give a proof of the verification theorem with boundary conditions given in Section 7 of the notes (Section 3).

2. Give an example of a deterministic optimal control problem with a boundary condition that has a smooth value function that so that $u(x,t) \neq g(x)$ for a large part of $x \in \Gamma$. Hint: the optimal control starting from points very close to x inside D should carry $X(t)$ away from x, probably to find a better f on some other part of the boundary.

3. Let $X(t) \in [-1, 1]$ satisfy

$$dX = \alpha(t)dt + \epsilon dW(t) ,$$

where $\alpha(t)$ is a control that may not depend on the future of t and $|\alpha| \leq 1$ for all t. Let τ be the hitting time $\tau = \min\{t \text{ with } |X(\tau)| = 1\}$. Consider the value function

$$u(x) = \min_{\alpha} E_x[\tau] .$$

As usual, the notation $E_x[\cdot]$ means that $X(0) = x$.

   (a) Write the Hamilton Jacobi Bellman equation for this problem that should be satisfied for $x \in (-1, 1)$.

   (b) What are the boundary conditions at $x = -1$ and $x = 1$?

   (c) Calculate the solution.

   (d) Verify from the foumula in part (c) that this solution converges to $u(x) = 1 - |x|$ as $\epsilon \to 0$.

4. Modify the optimal execution model in Problem 3 or Assignment 6 so that the stock price process is

$$dY(t) = -a\alpha Y dt + \sigma Y dW . \qquad (1)$$

The value function is

$$u(y, n) = \max_{\alpha} E_{y,n}\left[\int_0^{\infty} e^{-\rho t}\alpha(t)Y(t)\left(1 - \alpha(t)b\right) dt\right] .$$

Again $Y(0) = y$ and $N(0) = n$. The maximum is over all nonanticipating strategies.

   (a) Write the HJB equation for the value function, u.

(b) The PDE from part (a) involves first and second order derivatives. Show that the second order part is degenerate: the coefficient matrix of the matrix of second order derivatives is positive semi-definite but not positive definite.

(c) Show that the ansatz from Assignment 6, $u(y,n) = yv(n)$, reduces the two variable PDE to a single variable PDE. Write this one dimensional PDE for v.

(d) What does the PDE from part (c) say about the behavior of v and u as $n \to 0$? Is this to be expected?

(e) What does the PDE from part (c) say about the behavior of v and u as $n \to \infty$?