PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
Last corrected: April 25, 2008.

Assignment 8, due April 28

1. Let $f(s,t)$ be the solution of the vanilla option pricing problem. The option is a simple American style put with payout $(K-s)_+$. The risk neutral measure is geometric Brownian motion: $dS = fSdt + \sigma SdW$. The early exercise price is $B(t)$. The optimal strategy is to exercise the first time $S(t) = B(t)$ and to hold whenever $S(t) > B(t)$. Set $g(s,t) = \partial_t f(s,t)$. Show that $g(s,t)$ satisfies a Stefan type free boundary value problem with $\partial_t g + \frac{\sigma^2 s^2}{2}\partial_s^2 g + rs\partial_s g - rg = 0$ for $s > B(t)$ and two boundary conditions

$$g(B(t),t) \; = \; 0 \; , \tag{1}$$

and

$$\partial_s g(B(t),t) \; - \; \frac{2K}{B(t)^2\sigma^2}\,\dot{B} \; = \; 0 \; . \tag{2}$$

Hint: The smooth pasting condition is $\partial_s f(B(t),t) = -1$. Take $\frac{d}{dt}$ of both sides. The right gives zero. For the left, use the chain rule, as in:

$$\frac{d}{dt}\left(K - B(t)\right) \; = \; \frac{d}{dt} f(B(t),t) \; = \; \partial_s f(B(t),t)\dot{B} \; + \; \partial_t f \; .$$

This also helps derive (1). The equations (1) and (2) are much like two boundary conditions of the classical Stefan problem. One is a condition on the solution (e.g. temperature is zero). The other is a relation between the rate of movement of the boundary, $\dot{B}$, and the heat flux, $\partial_s g$.

2. This is a multi-asset version of one of the Merton optimal allocation problems. Suppose there are n risky assets, $S_k(t)$, that are geometric Brownian motions:

$$dS_k \; = \; \mu_k S_k dt \; + \; \sigma_k S_k dW_k \; . \tag{3}$$

Suppose the Brownian motions have correlations

$$\mathrm{corr}\left(W_j, W_k\right) \; = \; \rho_{jk} \; . \tag{4}$$

The matrix of correlations is ρ. It is positive definite. The diagonal entries are $\rho_{kk} = \mathrm{corr}\left(W_k, W_k\right) = 1$. The correlation between the Brownian motions does not change with time (a bad assumption in real markets, but a fine one for academic exercises). Let S_0 represent "cash" that earns interest rate r with no risk. The optimal allocation problem has allocation wieghts $\alpha_0(t)$, ..., $\alpha_n(t)$, that satisfy

$$\sum_{k=0}^{n} \alpha_k(t) \; = \; 1 \; ,$$

for all t. The value of the portfolio at time t is $X(t)$.

(a) Assuming nothing about the weights except that they are non-anticipating, write an expression for dX in terms of the α_k. This has the form

$$dX \;=\; \alpha_0(t)X(t)rdt + \cdots \;=\; rXdt + \sum_{k=1}^{n} (\mu_k - r)\,\alpha_k(t)X(t)dt + \cdots \;.$$

(b) Define the value function as the optimum

$$f(x,t) \;=\; E_{x,t}\left[\,\max_{\alpha[t,T]} U(X(T))\,\right]$$

Write an expression for df and then for $E\,[\,df \mid \mathcal{F}_t\,]$ in terms of $X(t)$ and the $\alpha_k(t)$.

(c) Solve the optimization problem at time t for the optimal α_k in terms of X, t, and f (and its derivatives). Show that the there is a $w(t)$ and numbers $\overline{\alpha}_k$ (for $k = 1, \ldots, n$, i.e. not $k = 0$) so that

$$\alpha_k(t) \;=\; w(t)\overline{\alpha}_k \quad , \qquad \sum_{k=1}^{n} \overline{\alpha}_k \;=\; 1\;. \tag{5}$$

The $\overline{\alpha}_k$ do not depend on t or on f or U, only on the parameters ρ, r, and μ_k. Of course, $\alpha_0(t) = 1 - w(t)$. There is no constraint that $\alpha_k(t) \geq 0$.

(d) Why is this called a *mutual fund theorem*? How is it related to simpler mutual fund theorems in the mean-variance theory of investments?

(e) Show that if $U(x) = x^\gamma$ for $0 < \gamma < 1$, then $w(t)$ is independent of t.

(f) Show that the answer to part (e) has the property that $w \to \infty$ and $\alpha_0 \to -\infty$ as $\gamma \to 1$. Use this to make a comment on the difference between maximizing expected value and maximizing a concave utility function.

3. It has been proposed that companies evaluate large capital intensive projects using *real option* theory. This theory emphasizes the option that the company has to discontinue the project if business conditions change. Suppose that a capital project requires fixed, known, constant capital rate M, which means that it requires $(t_2 - t_1)M$ during the time period (t_1, t_2) The completion date for the project is T. At time T, the company will receive a cash flow $S(T)$. The process $S(t)$ is observable and satisfies $dS = \mu S dt + \sigma S dW$. If the project is cancelled before time $\tau < T$, the total expense is τM and the income is zero. We want to choose τ to maximize the expected total revenue.

   (a) Define the appropriate value function. What variables does it depend on?

   (b) What PDE does it satisfy in the continuation region?

(c) What is the behavior of the early termination boundary as $t \to T$?

(d) What is the analogue of the smooth pasting condition at the early termination boundary? What is the derivative of the value function at the early termination boundary?

(e) How would you use this to advise the company on whether to start the project at all? (Hint: there is a relation between $S(0)$, μ, σ, T, etc. that should be satisfied.)