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Section 4: Some solution techniques

1 Affine models

Affine models are pricing equations in finance that have particular exponential solutions. The Heston stochastic volatility model is a famous example. Pricing problems for these models have semi-analytic solutions that take the form of an explicit formula for the Fourier transform. There does not seem to be an explicit formula for the Fourier integral, but even so, the FFT makes it possible to evaluate the solution quickly and accurately. The book *The Volatility Surface* by Jim Gatheral has more background and details than are presented here. And much of what's here comes from there.

1.1 The mathematical idea

The basic idea goes back at least to the great Russian physicist L. D. Landau. A linear PDE with constant coefficients may be solved by the Fourier transform because it has solutions that are plane waves. Landau observed that variable coefficient linear PDEs also may have plane wave solutions, but the wave vector will be time dependent. As a first model problem, take a linear advection equation with linear velocity field:

$$\partial_t u \;-\; \alpha \partial_x (xu) \;=\; 0 \;. \tag{1}$$

We have solved the initial value problem for this equation before, so we know that if the initial data form a plane wave, $u(x,0) = e^{ipx}$, then the solution has the form

$$u(x,t) \;=\; A(t) e^{ip(t)x} \;. \tag{2}$$

Before, we derived (2) from (1) using the method of characteristics. Now we do it a different way that illustrates the general method. Assume that (1) has a solution of the form (2) and seek formulas for p and A that make it work. In other words, we make (2) an *Ansatz* for the solution of (1). Rewrite the equation in characteristic form

$$\partial_t u \;-\; \alpha x \partial_x u \;-\; \alpha u \;=\; 0 \;,$$

and plug in $\partial_t u = \dot{A} e^{ipx} + i\dot{p}xAe^{ipx}$ and $\partial_x u = ipAe^{ipx}$, to get

$$\dot{A}e^{ipx} \;+\; i\dot{p}xAe^{ipx} \;+\; \alpha x i p A e^{ipx} \;+\; \alpha A e^{ipx} \;=\; 0 \;.$$

First we cancel the common exponential factor:

$$\dot{A} + i\dot{p}xA - \alpha xipA - \alpha A = 0 .$$

Then we set the coefficient of x and the coefficient of the constant term equal to zero separately, which gives

$$\dot{p} = \alpha p0 , \qquad \dot{A} = \alpha A . \tag{3}$$

Therefore,

$$p(t) = e^{\alpha t}p(0) , \quad A(t) = A(0)e^{\alpha t} .$$

Thus we learn that if $u(x,0) = e^{ipx}$, then

$$u(x,t) = e^{\alpha t}\, e^{ie^{\alpha t}px} . \tag{4}$$

First verify that the solution formula (4) agrees with what the solution is supposed to look like. Look at the imaginary part, $e^{\alpha t}\sin(e^{\alpha t}x)$. If $\alpha < 0$, the characteristics diverge, carrying the points where $u = 0$ away from the origin. The zeros of $\sin(e^{\alpha t}x)$ do spread out as they should. We also recognize the exponential decay of the amplitude of the solution in the case of diverging characteristics. Finally, the solution formula is somewhat complicated. This will get worse as we consider more complicated models.

The general initial data may be written as an integral superposition of plane waves

$$u(x,0) = f(x) = \int_{-\infty}^{\infty} e^{ipx}\, \widehat{f}(p)\, dp .$$

Since (4) says what happens to e^{ipx}, so we have

$$u(x,t) = \int_{-\infty}^{\infty} e^{\alpha t}\, e^{ie^{\alpha t}px}\, \widehat{f}(p)\, dp .$$

In complicated models, we will stop here, but for this case we can continue. Change variables in the integral: $e^{\alpha t}p = q$:

$$u(x,t) = \int_{-\infty}^{\infty} e^{iqx}\, \widehat{f}(e^{-\alpha t}q)\, dq . \tag{5}$$

Recall that the Fourier transform of $f(\lambda x)$ is $\frac{1}{\lambda}\widehat{f}(p/\lambda)$ and take $1/\lambda = e^{-\alpha t}$ and we see that $\widehat{f}(e^{-\alpha t}q)$ is the Fourier transform of $e^{\alpha t}f(e^{\alpha t}x)$. Therefore, the integral (5) is

$$u(x,t) = e^{\alpha t}f(e^{\alpha t}x) .$$

Please check that this formula satisfies the original advection equation (1). You found the same result in Homework 2 using the method of characteristics. Note that if $\alpha > 0$ and the characteristics are converging, the solution at x at time t is determined by the initial data at point $e^{\alpha t}x$, which is further away as it should be. The amplitude is determined by the condition that $\int u(x,t)dx = \int f(x)dx$.

The same method applies when we add a diffusion term to (1):

$$\partial_t u \;-\; \alpha x \partial_x u \;-\; \alpha u \;=\; \frac{1}{2}\partial_x^2 u \;. \tag{6}$$

The ansatz (2) is the same and the parameter equations (3) become

$$\dot{p} \;=\; \alpha p \;, \qquad \dot{A} \;=\; \alpha A \;-\; \frac{1}{2}p^2 A \;. \tag{7}$$

Again $p(t) = p(0)e^{\alpha t}$, which gives

$$\dot{A} \;=\; \left(\alpha \;-\; \frac{e^{2\alpha t}\, p(0)^2}{2}\right) A \;.$$

The solution is

$$A(t) \;=\; e^{\alpha t} \exp\!\left(\frac{-\left(e^{2\alpha t}-1\right) p(0)^2}{4\alpha}\right) \; A(0) \;.$$

One use of this formula is to write the general solution in terms of the Fourier transform of the initial data. We write p for $p(0)$ and $\widehat{f}(p)$ for $A(0)$:

$$u(x,t) \;=\; \frac{e^{\alpha t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipe^{\alpha t}x} \exp\!\left(\frac{-\left(e^{2\alpha t}-1\right) p^2}{4\alpha}\right) \; \widehat{f}(p)\, dp \;.$$

This is not quite in the form of a Fourier integral since it has $e^{ipe^{-\alpha t}x}$ instead of e^{ipx}. This can be fixed as above using the substitution $q = e^{\alpha t}p$:

$$u(x,t) \;=\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iqx} \exp\!\left(\frac{-\left(1-e^{-2\alpha t}\right) q^2}{4\alpha}\right) \; \widehat{f}(e^{-\alpha t}q)\, dq \;. \tag{8}$$

Observe that this formula states that the Fourier transform of $u(x,t)$ in the x variable is

$$\widehat{u}(p,t) \;=\; \exp\!\left(\frac{-\left(1-e^{-2\alpha t}\right) p^2}{4\alpha}\right) \; \widehat{f}(e^{-\alpha t}p) \;. \tag{9}$$

In particular, $\widehat{u}(0,t) = \widehat{f}(0)$, which indicates that $\int u(x,t)dx = \int f(x)dx$, as it should.

These formulas are getting complicated and will get more complicated. What use are they? One use finding a formula (also complicated) for the Green's function. The Green's function satisfies the initial value problem with initial data $f(x) = \delta(x-y)$, with y a fixed parameter. The solution is $G(y,x,t)$. Since we may write any initial data as $f(x) = \int f(y)\delta(x-y)dy$, we get a different formula for the solution of the initial value problem (review of Section 1)

$$u(x,t) \;=\; \int f(y) G(y,x,t)\, dy \;.$$

The Fourier transform of $\delta(x-y)$ is

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ipx} \delta(x-y)\, dx \;=\; \frac{e^{-ipy}}{\sqrt{2\pi}} \;.$$

Using this in (9) gives

$$\begin{aligned} G(y,x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} \exp\!\left(\frac{-\left(1-e^{-2\alpha t}\right)p^2}{4\alpha}\right) e^{-ipe^{-\alpha t}y}\, dp \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip(x-e^{-\alpha t}y)} \exp\!\left(\frac{-\left(1-e^{-2\alpha t}\right)}{2\alpha}\cdot\frac{p^2}{2}\right) dp \;. \end{aligned}$$

This might look like a mess, but the integral is over p and the exponential in the integrand has is just a linear plus a quadratic function of p. There is an explicit formula for such an integral, which we know as the Fourier transform of a Gaussian:

$$\int_{-\infty}^{\infty} e^{i\beta t - \frac{\gamma^2}{2}t^2}\, dt \;=\; \frac{\sqrt{2\pi}}{\gamma}\, e^{-\beta^2/2\gamma} \;.$$

Setting $t=p$, $\beta = x - e^{-\alpha t}y$, and $\gamma = \sqrt{\dfrac{1-e^{-2\alpha t}}{2\alpha}}$, gives

$$G(y,x,t) \;=\; \sqrt{\frac{\alpha}{\pi\left(1-e^{-2\alpha t}\right)}}\; e^{-\left(x-e^{-\alpha t}y\right)^2\alpha/\left(1-e^{-2\alpha t}\right)} \;. \tag{10}$$

As a function of x, this is a Gaussian with mean $e^{-\alpha t}y$. The variance converges to $1/2\alpha$ as $t\to\infty$. You may remember from Stochastic Calculus that the probability density of a particle satisfying the Ornstein Uhlenbeck process $dX = -\alpha X dt + dW$ converges to that same Gaussian as $t\to\infty$.

Integral formulas like (8) also are good for finding approximations to the solution in special cases. Such approximations may be more useful than complicated formulas. Look at (8) for large t. The integrand in (8) decays exponentially unless $\widehat{f}$ misbehaves. Therefore, most of the answer is determined by not too large q values. If t is large, $\widehat{f}(e^{-\alpha t}q) \approx \widehat{f}(0)$. Also, the real exponential factor is approximately $e^{-q^2/4\alpha}$. Therefore,

$$u(x,t) \;\approx\; \frac{1}{\sqrt{2\pi}}\widehat{f}(0)\int_{-\infty}^{\infty} e^{iqx} e^{-q^2/4\alpha}\, dq \;.$$

But $\widehat{f}(0) = \frac{1}{\sqrt{2\pi}}\int f(x)dx$, and the remaining integral is $\sqrt{\pi/4\alpha}\, e^{-\alpha x^2}$, so

$$u(x,t) \;\approx\; \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\cdot \int_{-\infty}^{\infty} f(y)\, dy \;.$$

This shows that the solution of the initial value problem for (6) has

$$u(x,t) \;\to\; \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}\cdot \int_{-\infty}^{\infty} f(y)\, dy \qquad \text{as } t\to\infty \;.$$

You should check that $e^{-\alpha x^2}$ is a steady solution (a solution with $\partial_t u = 0$) because it satisfies $\alpha \partial_x u + \alpha u = \frac{1}{2}\partial_x^2 u$. The constants $\sqrt{\alpha/\pi}$ and $\int f(y)dy$ are determined by conservation of mass. You probably learned in Stochastic Calculus that this is supposed to happen. The large time behavior of u depends only on the initial mass. It forgets everything else about the initial data.

1.2 Other exponential solutions

The exponential ansatz (2) and the algebra around it apply also to complex valued functions $p(t)$. The corresponding solutions are quite different despite the similarity in the algebra. The formula (10) suggests that we put quadratic terms as well as linear ones in the exponent. For example, we could seek a solution of (6) of the form

$$u(x,t) \;=\; A(t)e^{b(t)x} \;.$$

(Note the change in notation that signals the changing nature of the solution – the exponent is not a wave vector so we don't call it p.) The equation (6) becomes (dropping the t dependence for simplicity)

$$\dot{A}e^{bx} \;+\; A\dot{b}xe^{bx} \;-\; \alpha xAbe^{bx} \;-\; \alpha Ae^{bx} \;=\; \frac{1}{2}Ab^2e^{bx} \;.$$

We drop the common term e^{bx}, then equate coefficients of x, then the constant (in x) terms:

$$\dot{b} \;=\; \alpha b \;, \quad \dot{A} \;=\; \alpha A \;+\; \frac{Ab^2}{2} \;.$$

Of course, this is the same as (7) with the substitution $p = ib$. You should check that you understand why the exponential $e^{b(t)x}$ becomes steeper with time (hint: $\alpha > 0$ implies compression) and why the overall amplitude grows as well (diffusion).

The motivating formula (10) has a quadratic in the exponent. If we take such an ansatz:

$$u(x,t \;=\; A(t)e^{b(t)x+c(t)x^2/2} \;, \tag{11}$$

then (6) becomes (leaving out the common exponential factor)

$$\dot{A} \;+\; \dot{b}Ax \;+\; \dot{c}A\frac{1}{2}x^2 \;-\; \alpha xb \;-\; \alpha x^2c \;-\; \alpha a \;=\; \frac{1}{2}c^2x^2A \;+\; bcxA \;+\; \frac{1}{2}\left(b^2+c\right)A \;.$$

We collect the coefficients of x^2 to get:

$$\dot{c} \;=\; \alpha c \;+\; \frac{1}{2}c^2 \;. \tag{12}$$

This implies that if $c(0) > 0$, then c *blows up* at some time $T > 0$. This means that $c(t) \to \infty$ as $t \to T$. It indicates that there is no solution of this form beyond that point. It does not contradict the existence theorem because the existence theorem depends on some bound on how the initial data grow at infinity. Initial data like $e^{c(0)x^2}$ grows too fast.

On the other hand, if $c(0) < 0$ then the solution of (12) converges to a *fixed point*, c^*, which is a value of c for which $\cot c = 0$. This is $\alpha c + \frac{1}{2}c^2 = 0$, or $c^* = -2\alpha$. Of course, $1/c^* = 1/2\alpha$ is the asymptotic variance that was clear already in (10). We can find the solution corresponding to the Green's function by looking for $c \to -\infty$ as $t \to 0$. See Problem 2 of Homework 4.

1.3 The Heston model

The Heston model produces a theoretical price, $u(s, v, t)$, for a European style option that expires at time t from now assuming that the current (spot) price is s, the spot volatility is v, and the option payout is $f(s_t)$. The PDE is

$$\partial_t u \;=\; \frac{vs^2}{2}\partial_s^2 u \;+\; rs\partial_x u \;+\; ru \;+\; \rho\eta vs\partial_v\partial_s u \;+\; \frac{1}{2}\eta^2 v\partial_v^2 u \;+\; \lambda(v-\overline{v})\partial_v u \;. \tag{13}$$

We will talk about the derivation later, but for now, v represents the instantaneous squared volatility and s the spot price. They satisfy

$$\begin{aligned} dS(t) &= \mu s(t)dt \;+\; \sqrt{V(t)}S(t)dW_1(t) \\ dV(t) &= -\lambda(V(t)-\overline{v})dt + \eta\sqrt{V(t)}dW_2(t) \;, \end{aligned}$$

where W_1 and W_2 are Brownian motion paths with correlation coefficient ρ. We use the convention of representing random quantities by capital letters and the variables themselves (whatever that means) by lower case. As usual, μ is the (constant) rate of expected return, η is the volatility of volatility (volvol), $\overline{v}$ is the equilibrium squared volatility, and λ is the rate of memory loss for the fluctuating volatility. See Jim Gatheral's book.

We do some routine reductions before getting to the main point. Substituting $x = \ln(s)$ gives

$$\partial_t u \;=\; \frac{v}{2}\partial_x^2 u \;+\; (r - \frac{v}{2})\partial_x u \;+\; ru \;+\; \rho\eta v\partial_v\partial_x u \;+\; \frac{1}{2}\eta^2 v\partial_v^2 u \;+\; \lambda(v-\overline{v})\partial_v u \;.$$

The substitution $u = e^{rt}\widetilde{u}$ removes the term ru. The change of variable $x = \widetilde{x} - rt$ removes the $r\partial_x u$ term. What remains is a little smaller:

$$\partial_t u \;=\; \frac{v}{2}\partial_x^2 u \;-\; \frac{v}{2}\partial_x u \;+\; \rho\eta v\partial_v\partial_x u \;+\; \frac{1}{2}\eta^2 v\partial_v^2 u \;+\; \lambda(v-\overline{v})\partial_v u \;.$$

If we had all day (or all week as Heston must have had), we could solve this equation using the Fourier transform in x and v space. The first step would be to find solutions of the form

$$u(x,t) \;=\; A(t)e^{ip(t)x+iq(t)v} \;.$$

Unfortunately, this path leads to complex (in both senses) and confusing algebra that we don't have time to pick our way through.

Instead there is a simpler method for the most important case where the payout is independent of v (as most payouts are). This has the consequence that the initial data, which is the payout, are independent of v: $u(x,v,0) = f(x)$. None of the coefficients depends on x so we may take the Fourier transform in x only, to get $\widehat{u}(p,v,t)$, which satisfies

$$\partial_t \widehat{u} \;=\; \frac{-vp^2}{2}\widehat{u} \;-\; \frac{ivp}{2}\widehat{u} \;+\; \frac{1}{2}\eta^2 v \partial_v^2 \widehat{u} \;+\; \big(i\rho\eta vp \;+\; \lambda(v-\overline{v})\big)\,\partial_v \widehat{u}\;.$$

This is a PDE in one variable with coefficients proportional to that variable and p as a parameter. The initial condition is a constant (that depends on p but not v). Therefore, we seek a solution of the form

$$\widehat{u}(v,t;p) \;=\; A(t)e^{D(t)v}\;,$$

with $D(0) = 0$. The technique is as above but the algebra is more complicated. There is no closed form formula for the resulting Fourier integral $u(x,v,t) = \int e^{ipx}\widehat{u}(p,v,t)dp$. It is done in practice using the FFT.