PDE in Finance, Spring 2008,
`http://www.math.nyu.edu/faculty/goodman/teaching/PDEfin/index.html`
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Section 5: Probability fluxes

This section is very short since Kohn's notes on this material are excellent and detailed. The one thing I want to document is probability fluxes in the forward equation. These are important in Problem 3 of Homework 5.

For SDE $dX = a(X)dt + b(X)dW$, the forward equation is

$$\partial_t p \;=\; \frac{1}{2}\partial_{x_j}\partial_{x_k}\left(bb^t_{jk}(x)p\right) \;-\; \partial_{x_j}\left(\,a_j(x)p\,\right)\;. \qquad (1)$$

From this we can identify the probability flux vector

$$F_j(x,t) \;=\; -\sum_k \partial_k\left(bb^t_{jk}(x)p(x,t)\right) \;+\; a_j(x)p(x,t)\;. \qquad (2)$$

If $P_V(t) = \Pr(X(t) \in V)$, then

$$\frac{d}{dt}P_V(t) \;=\; \int_\Gamma F(x,t)n(x)\,dA(x)\;.$$

We do not give a derivation of (1), though that would be possible in a few pages. Instead, here are a few comments. The advective part of the flux is $F_{ad} = ap$. This is what it would be if there were no random forcing. Therefore, our understanding of it from previous sections still holds.

The diffusive flux has properties that may seem surprising at first, particularly if the diffusion coefficient bb^* is not constant. We think of diffusion as a way of evening out density or temperature, but $p = const$ does not give $F = 0$ or $\partial_t p = 0$. For example, in one dimension and with $a = 0$, the condition for steady state is $\partial_x(b^2(x)p) = 0$. So if at time $t = 0$ the probability density is uniform, then it will start moving, and moving toward regions where b^2 is smaller. Think of it this way. b^2 being small means not moving much. If you wander into those regions, you don't move (much) so you stay longer.