Risk and Portfolio Management with Econometrics, Courant Institute, Spring 2009
http://www.math.nyu.edu/faculty/goodman/teaching/RPME09/index.html
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Assignment 1, due January 28

Corrections: (none yet)

1. Study the consequences of mean variance analysis in the case of n uncorrelated assets. Assume that there are upper and lower bounds for the expected returns and the variances:

$$r < a \le \mu_i \le b \quad , \quad 0 < c \le \sigma_i^2 \le d \quad \text{for all } i. \tag{1}$$

   Assume that the market also has a risk free asset that has return r. Show that the Sharpe ratio goes to infinity as $n \to \infty$. More specifically, show that $\text{SR} \ge C\sqrt{n}$. The more uncorrelated assets you have to invest in, the less risk risk you need to achieve a given expected return. This is an illustration of *diversification*.

2. Redo Problem 1 under a one factor correlation model. In this model, there is a vector β with components β_i so that $\sigma_{ij} = \beta_i \beta_j$ for $i \ne j$. In addition to the bounds (1), assume the bounds

$$e \le \beta_i \le f \quad \text{for all } i. \tag{2}$$

   For this problem, assume that $e > 0$, although $\beta_i < 0$ is possible (though rare) in real markets. Assume that the diagonal elements of Σ have the form

$$\sigma_{ii} = \sigma_i^2 = \sigma_{D,i}^2 + \beta_i^2 , \tag{3}$$

   so that the correlations do not exceed unity: $\beta_i \beta_j \le \sigma_i \sigma_j$ for all $i \ne j$. Carry out the following sequence of steps:

   (a) Show that (3) implies that the correlation between R_i and R_j is less than unity. The correlation coefficient is $\rho_{ij} = \sigma_{ij}/(\sigma_i \sigma_j)$.

   (b) Show that variances and covariances described above would be realized in the model

$$R_i = \mu_i + R_{D,i} + \beta_i R_0 ,$$

   where the *diagonal* factors $R_{D,i}$ and the *market factor*, R_0 are all independent random variables, and $\text{var}[R_0] = 1$. The factors $R_{D,i}$ are usually called *idiosyncratic* risk factors, but we cannot denote this by $R_{i,i}$.

(c) The *rank* of a matrix, B, is the dimension of the subspace of R^n spanned by the columns of B. Show that B has rank one if and only if there are column vectors u and v so that $B = uv^t$. You may use the fact that the column rank of B (what is given above) is equal to the row rank (the dimension of the subspace spanned by the rows of B.

(d) Show that under the assumption above, that the covariance matrix has the form $\Sigma = \Sigma_D + \beta\beta^t$, where Σ_D is the diagonal matrix whose nonzero entries are the $\sigma^2_{D,i}$. Thus, show that Σ is a perturbation of a diagonal Σ_D by a matrix of rank one.

(e) Prove the *Sherman Morrison* formula

$$\left(A + uv^t \right)^{-1} = A^{-1} - \frac{1}{1+\gamma} A^{-1} uv^t A^{-1} \; , \tag{4}$$

where

$$\gamma = v^t A u \; . \tag{5}$$

Of course, you should assume that A^{-1} exists and that $\gamma \neq -1$. Hint: if you have square matrices B and C you can show $B = C^{-1}$ by showing that $BC = I$ (or $CB = I$, if that's easier).

(f) Use the Sherman Morrison formula to write a formula for the investment weights with covariance matrix Σ, in the case where the market does have a risk free asset.

(g) Show that in the limit $n \to \infty$, the weight on asset i depends only on β_i and not on the idiosyncratic risk $\sigma^2_{D,i}$. This is one of the main conclusions of the Markowitz theory.

(h) Show that the Sharpe ratio does not go to infinity as $n \to \infty$. You may do this by finding an explicit formula for the ratio and seeing what it does when n is large.

(i) Explain the result of Part (h) above by showing that the portfolio has a nonzero weight on the "market asset" R_0 that does not go to zero as $n \to \infty$.

(j) (zero credit, do not hand in, do only if time permits) Derive the Sherman Morrison formula by finding the coefficients in the Taylor series for $\left(A + tuv^t\right)^{-1}$ as a function of t expanded about $t = 0$.

3. Verify the Sherman Morrison formula using Matlab. Create a random $n \times n$ matrix A and two random vectors u and v. Form the left side of (4) using the Matlab matrix inverse function. Compute the right side directly and see that the two results are equal. Generate random matrices and vectors using the Matlab `randn()` function. Hand in printouts of the Matlab code and some results. You will be graded on following the following elements of good programming practice:

(a) Never *hard wire* a parameter like n. Instead, set the value `n = ...;` in some statement and have the rest of the code use n. The alternative (hard wiring) would be to choose to use, say, $n = 4$ throughout and just use 4 wherever n is needed. The problem with this is that if you then decide to try $n = 5$, you have to change many lines of code and you stand a fair chance of missing one.

(b) Use the Matlab *sprintf* command and other Matlab output formatting capabilities to make presentable printouts. The top line should say something like: `Testing the Sherman Morrison Formula with n = ....` Then print the matrices: `A = ..., u = ..., v = ...` and so on.

4. Verify the statement made in the notes that the same efficient portfolios and two fund theorem may be found by maximizing expected return with the given risk (variance of the return), and the same wealth constraint. Formulate this as a constrained optimization problem and express the solution using Lagrange multipliers. Call the Lagrange multipliers for the two constraints ν_1 and ν_2. Show that the set of portfolios as a function of nu_1 and ν_2 is the same as the set of portfolios as a function of λ_1 and λ_2 in the notes in the two fund theorem. Do this in the case where there is no risk free asset.