Risk and Portfolio Management with Econometrics, Courant Institute, Spring 2009
http://www.math.nyu.edu/faculty/goodman/teaching/RPME09/index.html
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Assignment 7, due April 8

Corrections: (none yet)

1. Let X be a random variable that represents the return on some investment portfolio that costs exactly one unit of currency today. The Sharpe ratio for X is
$$S_X \;=\; \frac{E\left[X - r\right]}{\sigma_X} \;,$$
where $\sigma_X = \sqrt{\text{var}(X)}$ is the standard deviation. Let p be a small probability and r a large (positive). The *lottery* is the random variable $Z = r$ with probability p and $Z = 0$ with probability $1 - p$. Suppose X is independent of Z and that $Y = X + Z$. Show that it is possible to choose r and p in such a way that $S_Y > S_X$. Hint: if p is very small and r is very large (with the right relationship) then σ_X differs from σ_Y by something that depends on $\sqrt{p}$. Of course, $E[X]$ differs from $E[Y]$ by something proportional to p, not $\sqrt{p}$. Some calculations and Taylor series arguments are involved.

2. Suppose the returns R_i are jointly normal with mean $E[R_i] = \mu_i$ and $\text{cov}[R_i, R_j] = \sigma_{ij}$. Let $R_w = \sum_i w_i R_i$ be the return corresponding to allocation w, with $\sum w_i = 1$. This is the framework of mean variance analysis, with the additional assumption that the returns are Gaussian. The efficient frontier is the set of portfolios so that you cannot increase $E[R_w]$ without also increasing σ_W. Let U be a strictly concave utility function ($U''(z) < 0$ for all z). Allocation w^* is optimal with respect to U if $E[U(R_{w^*})] \geq E[U(R_w)]$ for any other allocation, w. Show that if w^* is optimal for U then R_{w^*} is on the efficient frontier.

3. (Harder, do not spend too much time on this). Consider mean variance analysis in the case where the R_i are not Gaussian. Construct an example so that w^* is optimal with respect to some strictly concave utility function, but it is not on the efficient frontier. This shows that mean variance analysis can lead to "irrational" allocations in the non-Gaussian case. Hint: in the notation of Question 1, $E[U(Y)] > E[U(X)]$ for any $p > 0$ and $r > 0$. Now use the reasoning of Question 1.

4. This is a simplified version of the Merton optimal dynamic allocation problem. Suppose there is a time varying stock price that is a geometric Brownian motion $dS = \mu S dt + \sigma S dW$ (Don't worry if you don't know

stochastic calculus, this problem doesn't depend on it.). Suppose our wealth at time t is $Z(t)$, and it is allocated between the stock and a risk free asset that grown with rate r. In particular, suppose we fix an allocation ratio ρ so that $X = \rho Z$ is the stock allocation and $Y = (1-\rho)Z$ is the cash allocation. Here ρ is a fixed parameter, though the more complete Merton analysis allows ρ to be time dependent. We have

$$dZ \;=\; rZdt \,+\, (\mu - r)\rho Z dt + \sigma\rho Z dW \;.$$

The solution is

$$Z(T) \;=\; z_0 e^{rT} e^{(\mu-r)\rho T} e^{\sigma\rho W(T) - \frac{1}{2}\sigma^2\rho^2 T} \;.$$

Here z_0 is the value at time $t = 0$, and $W(t)$ is Gaussian with mean zero and variance t (That's what you need to know about Brownian motion and SDE to do this problem.). Suppose the utility function is $U(z) = z^\gamma$, with $0 < \gamma < 1$.

(a) Find a formula for the optimal ρ^* that maximized $E[(Z(T))]$. Show that this is independent of T but does depend on the other parameters.

(b) Show that $\rho^* \to \infty$ as $\gamma \to 1$. Note that $\rho > 1$ implies that the cash position is negative – borrowing to buy stock. Explain what this says about optimizing only $E[Z(T)]$ without considering risk.