Assignment 2.

Given February 3, due February 10.

Objective: To work with Taylor series in elementary numerical analysis.

1. Examine curve A and curve B. One of the functions has a jump in its third derivative while the other is completely smooth. Try to determine which curve has a discontinuous third derivative and locate the discontinuity on the graph. Do not spend much of time doing this in a quantitative way, but do make a guess. In the past, there have been more wrong than correct answers.

2. Extrapolation means estimating the value of a function from known values on one side only. For example, suppose $g(x)$ is a smooth function of x and we know the values $g(0)$, $g(h)$, $g(2h)$, and so on. We wish to estimate, say, $g(-h)$ or $g(-2h)$ from the known values. One instance would be the estimate:

$$g(-2h) \approx a_0 g(0) + a_1 g(h) + a_2 g(2h) + a_3 g(3h) \; . \tag{1}$$

The coefficients happen not to depend on h in this problem. Show that we get first order accurate estimates of $g(-h)$ or $g(-2h)$ using only $g(0)$, second order accuracy with $g(0)$, and $g(h)$, and so on to fourth order accuracy with (1). We say that g is *constant* if $g(x) = b_0$ for all x, *linear* if $g(x) = b_0 + b_1 x$, *quadratic* if $g(x) = b_0 + b_1 x + b_2 x^2$, etc. Show that these extrapolations are exact for constants, linears, quadratics, and cubics respectively.

3. The interval $(0, L)$ is divided into n subintervals, called *cells*, of equal length $\Delta x = L/n$. Use the notation $x_k = k\Delta x$ for the endpoints of the cells, and $I_k = (x_k, x_{k+1})$ for the cells themselves. It will be convenient to write $x_{k+1/2} = (k + 1/2)\Delta x$ for the midpoint of I_k. We do not have the values of f, but we do have the *cell averages*

$$F_k = \frac{1}{\Delta x} \int_{x_k}^{x_{k+1}} f(x) dx \; .$$

Assume that $f(x)$ is a smooth function. Show that $F_k = f(x_{k+1/2}) + O(\Delta x^2)$. Hint: it will be easier if you use a variable $y = x - x_{k+1/2}$ and perform Taylor series expansions about $x = x_{k+1/2}$, $y = 0$.

4. Find a second order accurate estimate of $f(x_k)$ in terms of cell averages. The estimate will be different for the *interior* points, x_1, …, x_{n-1} and the *boundary* points x_0 and x_n. For the interior points, find an approximation

$$f(x_k) = aF_{k-1} + bF_k + O(\Delta x^2) \; .$$

The estimate of $f(0)$ should use F_0 and F_1.

5. If we wish to use the interior estimation formula at $x = 0$, which would be $f(0) \approx aF_{-1} + bF_0$, we face the problem that F_{-1} is not known. We can circumvent this problem by extrapolating a value for F_{-1} using known values values F_0, F_1, etc. as in part 2. For example, suppose we use F_0, F_1, and F_2 to calculate $\widehat{F}_{-1} \approx F_{-1}$, then $f(0) \approx \widehat{f}_0 = a\widehat{F}_{-1} + bF_0$, the result will be an approximation $\widehat{f}_0 = cF_0 + dF_1 + eF_2$. What order of extrapolation do you need to get a second order estimate of $f(0)$? Is this estimate the same as the second order estimate from part 4?

6. Find a fourth order estimate of $f(x_{k+1/2})$ using F values. First find the interior formula

$$f(x_{k+1/2}) \approx aF_{k-1} + bF_k + cF_{k+1} \ ,$$

that works when $k > 0$ and $k < n$. Then, (as time permits) figure out what order of extrapolation and how many extrapolated values you need to achieve fourth order at the end points.