Assignment 4.

Given March 3, due March 10.

Objective: Numerical linear algebra.

1. Let $u = (u(1), u(2), \ldots, u(n))$ be a long sequence of numbers. We say that $u \in V$ if, for each integer k in the range $0 \leq k \leq n-3$, the relation

$$u(k+3) + 2u(k+2) - u(k+1) - 2u(k) = 0 \tag{1}$$

is satisfied. This is a theoretical exercise that goes through some basic definitions in linear algebra and at the same time shows how to understand solutions of *recurrence relations* like (1). When you are done, you will be able to make quantitiative predicions of the loss of accuracy in the Fibonacci and pib number computations you observed in the first assignment. No programming is required for this question.

A. Verify that V is a vector space by showing that if $u \in V$ and $v \in V$, then $u + v \in V$ and $xu \in V$ for any scalar, x.

B. Show that if $u(0)$, $u(1)$, and $u(2)$ are given, then all the rest of the $u(k)$ are determined by (1).

C. Here is a more complicated way of saying the same thing. Let v_0, v_1, and v_2 be the sequences in V given by

$$\begin{aligned} v_0 &= (1, 0, 0, 2, -4, \cdots) \\ v_1 &= (0, 1, 0, 1, 0, \cdots) \\ v_2 &= (0, 0, 1, -2, 5, , \cdots) \ . \end{aligned}$$

For example, $v_2(4) = -2$. Show that these form a basis for V. To be a basis, the vectors v_0, v_1, and v_2 must be linearly independent, and it must be possible to express any element of V as a linear combination of them. For the latter point, first demonstrate, then use the fact that that if $u \in V$, then

$$u(k) = u(0)v_0(k) + u(1)v_1(k) + u(2)v_2(k) \ .$$

D. A geometric sequence has the form $w(k) = z^k$ for some number, z. Show that there are three linearly independent geometric sequences in V and find the corresponding numbers z_1, z_2, and z_3. How do we know that these span V?

E. Define the matrix A_n by

$$\begin{pmatrix} u(n-2) \\ u(n-1) \\ u(n) \end{pmatrix} = A_n \begin{pmatrix} u(0) \\ u(11) \\ u(2) \end{pmatrix} \ .$$

Use the basis from part D to find more or less explicit matrices B, C so that $A_n = BZ^{n-2}C$, where Z is a diagonal matrix with the numbers z_1, z_2, and z_3 on the diagonals.

F. Show that $\|A_n\|$ and $\|A_n^{-1}\|$ both grow exponentially (in the literal sense) with n. What does this say about the condition number of a_n for large n?

F. Use this to explain the results of the Fibonacci and pib number parts of assignment 1.

2. Write a procedure in C/C++ to compute the Choleski decomposition of a positive definite symmetric matrix. Your procedure should take an $n \times n$ real symmetric matrix, A, and compute an $n \times n$ lower triangular matrix, L, with $LL^* = A$. It might be convenient to store only the lower triangular entries (including the diagonal) or it might be easier to store the whole matrix but assume that $a_{jk} = a_{kj}$. There are n square roots to take in this process. Your procedure should return an error flag if any of these is not real. Make sure also to write a procedure that computes LL^* from a lower triangular matrix L to check that your factor is correct. Apply your Choleski factorization procedure to the $n \times n$ matrix

$$A(s) = \begin{pmatrix} 2 & 1 & 1 & \cdots & & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 2 & 1 & \cdots & 1 \\ \vdots & & & \ddots & & \vdots \\ & & & & 2 & 1 \\ 1 & \cdots & & & 1 & 2 \end{pmatrix} - s \begin{pmatrix} 2 & 1 & 0 & \cdots & & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ & & & & 2 & 1 \\ 0 & \cdots & & & 1 & 2 \end{pmatrix}$$

Write a procedure to apply forward and back substitution on given Choleski factors and a tester to make sure the answer is correct. Use your Choleski and back substitution procedures to make a plot of $x_1(s)$ (the first component of $x(s)$ where $A(s)x(s) = e_1$ (the "standard" basis vector with one in the first entry and the rest zero). Starting from $s = 0$ and stopping at the first place where $A(s)$ is determined not to be positive definite. In your program, this means that we attempt to compute the Choleski factor of $A(s)$ and continue until the Choleski factorization procedure returns an error flag indicating that this is impossible. Take steps $\Delta s = .01$ and $n = 20$. *This involves considerable programming. Make sure to hand in both procedures and both testers, runs indicating the tests work, as well as the graphs requested. We will use this Choleski code in later assignments.*