Practice and Instructions for the Final Exam

Information and rules for the final

- The final exam is 4:00 to 4pm on Thursday, December 19.
- No electronic devices may be visible during the exam.
- The exam is “closed book”. No materials are allowed except for one “cheat sheet”, which is one standard piece of paper that you may prepare in advance with any information you want.
- Please explain all answers with a few words or sentences. You may lose points for a formula or answer with no explanation.
- You will get 25% credit for any question left blank. Points may be deducted for wrong answers, which can bring the credit to zero.
- Please cross out any answers you think are wrong. You may lose points for wrong answers even if you also give the right answer.
- Write answers in an exam book (“blue book”).

See also practice final exam questions from recent years.
Added a correction to Full Answer question 3 and fixed Multiple Choice (1e).

True/False

In each case, say whether the statement is true or false and give few words or sentences to justify your answer.

1. In IEEE format 64 bit floating point arithmetic, integers are represented exactly as long as they are within the range of floating point, which is up to around 10^{300}.
2. Multiplication in IEEE floating point is commutative in the sense that $x * y = y * x$ for any floating point numbers x and y, but it is not necessarily commutative with more than two numbers in the sense that $x * y * z \neq z * x * y$ is possible.
3. There is a way to code a Monte Carlo algorithm so that it produces the same results exactly every time it is run.
4. We want to approximate
$$A = \int_0^1 f(x)\, dx \; .$$
The integrand is a smooth function of x. Monte Carlo is likely to be the cheapest (least computer time, fewest evaluations of f) way to estimate A accurately.
5. Let $V(x_1, x_2, x_3)$ be a smooth function of three variables. Let H be the matrix
$$H = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$
Then $p = -H\nabla V(x)$ is a descent direction.
6. Suppose $A(h) = A + hA_1 + O(h^2)$, then $B(h) = 2A(h) - A(2h)$ is second order accurate.

7. Suppose $A(h) = A + O(h)$, then $B(h) = A(2h) - A(h)$ is second order accurate.

8. If $\widehat{A}$ is an approximation of A with accuracy $\left|\widehat{A} - A\right| \leq 10^{-3}$, then $\widehat{A}$ has a high relative accuracy as an approximation.

Multiple Choice

In each case, select the correct answer and justify your choice with a few words or sentences.

1. Rank the following from fastest to slowest, in running time using the Python codes given. Here, A is a two index numpy ndarray of size 1000×1000. Numpy arrays are normally stored in *row major* order, which means that matrix elements are stored as in this example:

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 8 & 9 & 10 & 11 \end{pmatrix}$$

(a)
```
n = 1000
sum = 0.
for i in range(n)
   sum += np.sum(A[:,i])  # a column at a time
```

(b)
```
n = 1000
sum = 0.
for i in range(n)
   sum += np.sum(A[i,:])  # a row at a time
```

(c)
```
n = 1000
sum = 0.
for i in range(n):
   for j in range(n):
      sum += np.sum(A[i,i])  # scalar double loop
```

(d)
```
n = 1000
sum = np.sum(A)
```

(e)
```
n = 1000
Av = np.reshape(A,n*n)  # put all of A in a one index array
v1 = np.ones([n*n])     # a vector with n^2 ones
sum = np.dot(Av,v1)     # complicated but correct
```

Full answer

1. Let $x(t)$ be a trajectory of an ODE system $\dot{x} = f(x)$ with $x(0) = x_0$ given. Give an algorithm to find

$$T_* = \min_{t>0} \text{ with } \|x(t)\|_2 = 1 \, .$$

The error should be second order in a time step parameter Δt.

2. Suppose $A(t)$ is an $n \times n$ matrix with entries that depend on t in a smooth way. Suppose the eigenvalues of A are all real even though A may not be symmetric. Find a formula for $\dot{\lambda}_1$ if the eigenvalues and eigenvectors themselves are already computed. You may use both right and left eigenvectors $A(t)r_j(t) = \lambda_j(t)r_j(t)$ and $\ell_j(t)A(t) = \lambda_j(t)\ell_j(t)$.

3. Consider the linear least squares problem with a matrix that is a differentiable function of a parameter t

$$u(t) = \min_x \|Ax - b\| \, .$$

Suppose $\dot{A}$ is the matrix of time derivatives of elements of A. Find a formula/algorithm for calculating $\dot{u}(t)$ that uses whatever factorization is used to compute x and also $\dot{A}$. This should be based on perturbation theory.

4. Show that one level of Richardson extrapolation takes the second order accurate trapezoid rule for integration to a fourth order integration rule. That is, if

$$A = \int_0^1 f(x)\,dx$$

$$\widehat{A}_2(h) = h\left[\frac{1}{2}f(0) + \sum_1^{n-1} f(x_k) + \frac{1}{2}f(1)\right]$$

Here, $\widehat{A}_2$ is a second order approximation, $nh = 1$, and $x_k = kh$.

(a) Find a formula $\widehat{A}_4(h) = \alpha\widehat{A}_2(h) + \beta\widehat{A}_2(2h)$. that has at least one order higher of accuracy.

(b) Show that for the trapezoid rule, your extrapolated approximation is fourth order accurate.

(c) Consider using the same α and β coefficients to improve the accuracy of a second order two stage Runge Kutta method such as predictor corrector trapezoid or midpoin. Explain why the resulting method is third order rather than fourth order.