Assignment 4.

Given February 26, due March 4.

Objective: To learn about the Singular Value Decomposition.

The singular value decomposition (SVD) is a matrix factorization. If A is an $n \times m$ matrix, then we may write A as a product of three factors:

$$A = U\Sigma V^* \quad , \tag{1}$$

where U is an orthogonal $n \times n$ matrix, V is an orthogonal $m \times m$ matrix, V^* is the transpose of V, and Σ is an $n \times m$ matrix that has all zeros except for its diagonal entries, which are nonnegative real numbers. If σ_{ij} is the i,j entry of Σ, then $\sigma_{ij} = 0$ unless $i = j$ and $\sigma_{ii} = \sigma_i \geq 0$. The σ_i are the "singular values" and the columns of u and v are respectively the right and left singular vectors. In these exercises, we always suppose that the singular values have been ordered so that

$$\sigma_1 \geq \sigma_2 \geq \cdots \quad .$$

This assignment does not ask you to touch a computer.

(1) Show that if the matrix A is square and symmetric then the singular values of A are the absolute values of the eigenvalues of A and that the singular vectors are eigenvectors of A.

(2) Show that $\|A\|_2 = \sigma_1$. If A is square and A^{-1} exists, then $\left\|A^{-1}\right\|_2 = \sigma_n^{-1}$. Show that the condition number of A (for the 2-norm) is σ_1/σ_n.

(3) "Regularization" is a way to compromise between accurate solution of a least squares problem and the size of the solution. Regularization is used to improve the conditioning of ill conditioned least squares problems. The simplest regularization strategy is to replace the least squares problem

$$\min_x \|Ax - b\|_2$$

with the "regularized" problem

$$\min_x \|Ax - b\|_2^2 + \epsilon \|x\|_2^2 \quad . \tag{2}$$

Give an algorithm to solve the regularized least squares problem, (2), using the SVD of A. Your algorithm will involve $1/(\sigma_i + \epsilon)$. Does this make dimensional sense?

(4) Use the SVD to show that the non zero eigenvalues of A^*A are identical to the non zero eigenvalues of AA^*. Note that these matrices have different dimensions so they have a different number of eigenvalues.

(5) The "polar decomposition" of a square matrix, A, is a decomposition $A = RQ$ where R is symmetric and positive semi-definite and Q is orthogonal. This is a generalization of the polar decomposition of a complex number $z = re^{i\theta}$ into a product of a non negative real number and a complex number of magnitude 1. The polar decomposition of matrices is used in elasticity theory. Show how to find the polar decomposition of a matrix from its SVD.