Stochastic Calculus Final Exam
December 19, 2002

- Answer all questions in the blue answer books.
- On each book, write your name and the number of books used. Number each book, e.g. "book 2 of 3".
- Cross out or erase all work you know to be wrong. If you have a right and a wrong answer, I will take off for the wrong answer as well as giving credit for the correct one.
- Explain and show all work, as time allows. I will take off points for formulas that are not explained.

1. In each case, indicate whether the statement is true or false and give a short explination of your answer, with either a reason for it to be true or a counterexample.

a. The random variables X and Y are jointly normal with mean zero, $\text{var}(X) = 1$, $\text{var}(Y) = 1$, and $\text{cov}(X, Y) = \frac{1}{2}$. Define $U = X - 2Y$. This U is uncorrelated with X, and independent of X.

b. With X and Y as above, define $V = X^2 + X - 2Y$. This V is uncorrelated with X and independent of X.

c. Let X_n be a discrete time martingale. Let $\mathcal{G}_n$ be the $\sigma-$algebra generated by X_n only (i.e. not X_{n-1}, etc.). Then $E[X_{n+1} \mid \mathcal{G}_n] = X_n$.

d. If Y_n has $E[Y_{n+1} \mid \mathcal{G}_n] = Y_n$, then Y_n is a martingale.

e. If $\mathcal{F}$ and $\mathcal{G}$ are two algebras of sets in the probability space Ω, then the intersection $\mathcal{F} \cap \mathcal{G}$ is also an algebra of sets.

f. If Q is a probability measure on the space of continuous paths $C([0, T] \to R)$ that is absolutely continuous with respect to Brownian motion measure (Wiener measure), then the total variation of a path X_t for $0 \leq t \leq T$ is infinite with Q probability one. More formally, if $A \subset C([0, T] \to R)$ is the set of paths with infinite total variation, then $Q(A) = 0$ if Q is absolutely continuous with respect to Wiener measure.

2. **a.** For a normal random variable $X \sim \mathcal{N}(0, \sigma^2)$, write the PDF for X and use it to calculate $E[X^2 e^X]$.

b. For X and Y as in question 1a, calculate $E[Y^2 e^X]$. Hint: express X and Y in terms of independent gaussians.

3. An n state Markov chain has transition matrix P. The state after t transitions is X_t, which is one of the values $(1, \ldots, n)$. We also have a Markov stopping time, $\tau \geq 0$, independent of the Markov chain. A Markov stopping time depends on the parameter, $q = \Pr(\tau > t \mid \tau \geq t)$.

a. Write an expression for $\Pr(\tau = t)$.

b. Find a matrix equation involving P for the τ step transition probabilities $v_{jk} = \Pr(X_\tau = k \mid X_0 = j)$. These v_{jk} form the τ step transition probability matrix V. Write a matrix equation involving V and P. Hint: Starting from state j, break the event $A_k = \{X_\tau = k\}$ into two sets, one with $\tau = 0$ ($Z = \{\tau = 0\}$) and one with $\tau > 0$ ($B = \{\tau > 0\}$). It is easy to calculate $\Pr(X_\tau = k \mid Z,\ X_0 = j)$. For the B part, we can calculate

$$\Pr(X_\tau = k \mid B,\ X_0 = j) = \sum_l \Pr(X_\tau = k \mid X_1 = l) \cdot \Pr(X_1 = l \mid B,\ X_0 = j) \ .$$

Because τ and X_t are Markov, the first factors in the sum on the right are entries of V.

4. Suppose $x = (x_1, x_2)^* \in R^2$ and $f = (f_1, f_2)^* \in R^2$. The * means transpose, so that the column vector x has $x^* = (x_1, x_2)$, which is a row vector. The component of f perpendicular to x is $g = f - (x^* f / x^* x)x = f - (x^* f / r^2) f$, where $r^2 = \|x\|^2 = x_1^2 + x_2^2$. The quantity in parentheses is a number, not a vector.

a. Show that $x^* g = 0$.

b. Show that for any $f(x)$, the system of two ordinary differential equations

$$\frac{dx}{dt} = g(x) = f(x) - \frac{x^* f(x)}{r^2} x$$

has $r^2 = \textit{const}$ by calculating

$$\frac{d}{dt} x^*(t) x(t) = 2x^* \frac{dx}{dt} \quad \text{or} \quad d(r^2) = d(x^* x) = 2x^* dx = 2x^* g dt \ .$$

c. Suppose we have the two dimensional Ito SDE

$$dX(t) = dW(t) - \frac{X(t)^* dW(t)}{R(t)^2} X(t) \quad \text{with} \quad R(t)^2 = X_1(t)^2 + X_2(t)^2 \ .$$

Here $X_1(t)$ and $X_2(t)$ refer to the first and second components of $X(t) \in R^2$. Show that $R(t)$ is not constant in time.

d. Find a formula for $R(t)$ in terms of $R(0)$ and t. Show that if $R(0)$ is known, then $R(t)$ is not random.

5. We have a system of two SDEs:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{V_t} S_t dW_t^{(1)} \ , \\ dV_t &= (a - bV_t)dt + \mu dW_t^{(2)} \ , \end{aligned}$$

where $W^{(1)}$ and $W^{(2)}$ are independent Brownian motions. The initial conditions for the SDE are $S_0 = 1$, $V_0 = .4$. We want to calculate

$$f = E\left[\int_0^T S_t^2 dt \right] \ .$$

Explain how to do this using a PDE. Give the PDE and whatever other information (possibly including initial data, final data boundary conditions, etc.) is needed. Make sure to give a probabilistic definition of the quantity that satisfies the PDE.