Assignment 5.

Given October 1, due October 21. Last revised, October 7.
Objective: Brownian Motion.

1. Suppose $h(x)$ has $h'(x) > 0$ for all x so that there is at most one x for each y so that $y = h(x)$. Consider the process $Y_t = h(X_t)$, where X_t is standard Brownian motion. Suppose the function $h(x)$ is smooth. The answers to the questions below depend at least on second derivatives of h.

a. With the notation $\Delta Y_t = Y_{t+\Delta t} - Y_t$, for a positive Δt, calculate $a(y)$ and $b(y)$ so that $E[\Delta Y_t \mid \mathcal{F}_t] = a(Y_t)\Delta t + O(\Delta t^2)$ and $E[\Delta Y_t^2 \mid \mathcal{F}_t] = b(Y_t)\Delta t + O(\Delta t^2)$.

b. With the notation $f(Y_t, t) = E[V(Y_T) \mid \mathcal{F}_t]$, find the backward equation satisfied by f. (Assume $T > t$.)

c. Writing $u(y, t)$ for the probability density of Y_t, *use the duality argument* to find the forward equation satisfied by u.

d. Write the forward and backward equations for the special case $Y_t = e^{cX_t}$. Note (for those who know) the similarity of the backward equation to the Black Scholes partial differential equation.

2. Use a calculation similar to the one we used in class to show that $Y_T = X_T^4 - 6\int_0^T X_t^2 dt$ is a martingale. Here X_t is Brownian motion.

3. Show that $Y_t = \cos(kX_t)e^{k^2 t/2}$ is a martingale.

a. Verify this directly by first calculating (as in problem 1) that

$$E[Y_{t+\Delta t} \mid \mathcal{F}_t] = Y_t + O(\Delta t^2) \; .$$

Then explain why this implies that Y_t is a martingale exactly (Hint: To show that $E[Y_{t'} \mid \mathcal{F}_t] = Y_t$, divide the time interval (t, t') into n small pieces and let $n \to \infty$.

b. Verify that Y_t is a martingale using the fact that a certain function satisfies the backward equation. Note that, for any function $V(x)$, $Z_t = E[V(X_T) \mid \mathcal{F}_t]$ is a martingale (the tower property). Functions like this Z satisfy backward equations.

c. Find a simple intuition that allows a supposed martingale to grow exponentially in time.

4. Let $A_{x_0,t}$ be the event that a standard Brownian motion starting at x_0 has $X_{t'} > 0$ for all t' between 0 and t. Here are two ways to verify the large time asymptotic approximation $P(A_{x_0,t}) \approx \frac{1}{\sqrt{2\pi}}\frac{2x_0}{\sqrt{t}}$.

a. Use the formula from "Images and reflections" to get

$$\begin{aligned} P(A_{x_0,t}) &= \int_0^\infty u(x,t)dx \\ &\approx \frac{1}{\sqrt{2\pi t}} \int_0^\infty e^{-x^2/2t}\left(e^{xx_0/t} - e^{-xx_0/t}\right) dx \; . \end{aligned}$$

The change of variables $y = x/\sqrt{t}$ should make it clear how to approximate the last integral for large t.

b. Use the same formula to get

$$\frac{-d}{dt} P(A_{x_0,t}) = \frac{1}{\sqrt{2\pi}} \frac{2x_0}{t^{3/2}} e^{-x_0^2/2t} \; . \tag{1}$$

Once we know that $P(A_{x_0,t}) \to 0$ as $t \to \infty$, we can estimate its value by integrating (1) from t to ∞ using the approximation $e^{const/t} \approx 1$ for large t. Note: There are other hitting problems for which $P(A_t)$ does not go to zero as $t \to \infty$. This method would not work for them.