Assignment 9.

Given December 9, due December 23. Last revised, December 91.

Instructions: Please answer these questions without discussing them with others or looking up the answers in books.

1. Let $\mathcal{S}$ be a finite state space for a Markov chain. Let $\xi(t) \in \mathcal{S}$ be the state of the chain at time t. The chain is *nondegenerate* if there is an n with $P^n_{jk} \neq 0$ for all $j \in \mathcal{S}$ and $k \in \mathcal{S}$. Here the P_{jk} are the $j \to k$ transition probabilities and P^n_{jk} is the (j,k) entry of P^n, which is the n step $j \to k$ transition probability. For any nondegenerate Markov chain with a finite state space, the *Perron Frobeneus theorem* gives the following information. There is a row vector, π, with $\sum_{k\in\mathcal{S}} \pi(k) = 1$ and $\pi(k) > 0$ for all $k \in \mathcal{S}$ (a probability vector) so that $\|P^t - \mathbf{1}\pi\| \leq Ce^{-\alpha t}$. Here $\mathbf{1}$ is the column vector of all ones and $\alpha > 0$. In the problems below, assume that the transition matrix P represents a nondegenerate Markov chain.

   (a) Show that if $P(\xi(t) = k) = \pi(k)$ for all $k \in \mathcal{S}$, then $P(\xi(t+1) = k) = \pi(k)$ for all $k \in \mathcal{S}$. In this sense, π represents the *steady state* or *invariant* probability distribution.

   (b) Show that P has one eigenvalue equal to one, which is simple, and that every other eigenvalue has $|\lambda| < 1$.

   (c) Let $u(k,t) = P(\xi(t) = k)$. Show that $u(k,t) \to \pi(k)$ as $t \to \infty$. No matter what probability distribution the Markov chain starts with, the probability distribution converges to the unique steady state distribution.

   (d) Suppose we have a function $f(k)$ defined for $k \in \mathcal{S}$ and that $E_\pi[f(\xi)] = 0$. Let f be the column vector with entries $f(k)$ and $\widehat{f}$ the row vector with entries $\widehat{(f)}(k) = f(k)\pi(k)$. Show that

   $$\mathrm{cov}_\pi(f(\xi(0)), f(\xi(t))) = E_\pi[f(\xi(0)), f(\xi(t))] = \widehat{f}P^t f \; .$$

   (e) Show that if A is a square matrix with $\|A\| < 1$, then

   $$\sum_{t=0}^{\infty} A^t = (I - A)^{-1} \; .$$

   This is a generalization of the geometric sequence formula $\sum_{t=0}^{\infty} a^t = 1/(1-a)$ if $|a| < 1$, and the proof/derivation can be almost the same, once the series is shown to converge.

   (f) Show that if $E_\pi[f(\xi)] = 0$, then $\sum_{t=0}^{\infty} P^t f = g$ with $g - Pg = f$ and $E_\pi[g(\xi)] = 0$. If the series converges, the argument above should apply.

(g) Show that

$$C = \sum_{t=0}^{\infty} \text{cov}_\pi[f(\xi(0)), f(\xi(t))] = \widehat{f}g \ ,$$

where g is as above.

(h) Let $X(T) = \sum_{t=0}^{T} f(\xi(t))$. Show that $\text{var}(X(T)) \approx DT$ for large T, where

$$D = \text{var}_\pi[f(\xi)] + 2\sum_{t=1}^{\infty} \text{cov}_\pi[f(\xi(0)), f(\xi(t))] \ .$$

This is a version of the Einstein Kubo formula. To be precise, $\frac{1}{T}\text{var}(X(T)) \to D$ as $T \to \infty$. Even more precisely, $|\text{var}(X(T)) - DT|$ is bounded as $T \to \infty$. Prove whichever of these you prefer.

(i) Suppose P represents a Markov chain with invariant probability distribution π and we want to know $\mu = E_\pi[f(\xi)]$. Show that $\widehat{\mu}_T = \frac{1}{T}\sum_{t=0}^{T} f(\xi(t))$ converges to μ as $T \to \infty$ in the sense that $E[(\widehat{\mu}_T - \mu)^2] \to 0$ as $T \to \infty$. Show that this convergence does not depend on $u(k,0)$, the initial probability distribution. It is not terribly hard (though not required in this assignment) to show that $\widehat{\mu}_T \to \mu$ as $T \to \infty$ almost surely. This is the basis of *Markov chain Monte Carlo*, which uses Markov chains to sample probability distributions, π, that cannot be sampled in any simpler way.

(j) Consider the Markov chain with state space $-L \leq k \leq L$ having $2L+1$ states. The one step transition probabilities are $\frac{1}{3}$ for any $k \to k-1$, $k \to k$ or $k \to k+1$ transitions that do not take the state out of the state space. Transitions that would go out of $\mathcal{S}$ are *rejected*, so that, for example, $P(L \to L) = \frac{2}{3}$. Take $f(k) = k$ and calculate π and D. Hint: the general solution to the equations $(g - Pg)(k) = k$ is a cubic polynomial in k.

2. A Brownian bridge is a Brownian motion, $X(t)$, with $X(0) = X(T) = 0$. Find an SDE satisfied by the Brownian bridge. Hint: Calculate $E_{x,t}[\Delta X \mid X(T) = 0]$, which is something about a multivariate normal.

3. Suppose stock prices $S_1(t), \ldots, S_n(t)$ satisfy the SDEs $dS_k(t) = \mu_k S_k dt + \sigma_k S_k dW_k(t)$, where the $W_k(t)$ are *correlated* standard Brownian motions woth correlation coefficients $\rho_{jk} = \text{corr}(W_j(t), W_k(t))$.

   (a) Write a formula for $S_1(t), \ldots, S_n(t)$ in terms of *independent* Brownian motions $B_1(t), \ldots, B_n(t)$. You may use the Cholesky decomposition $LL^t = \rho$.

   (b) Write a formula for $u(s,t)$, the joint density function of $S(t) \in R^n$. This is the n dimensional correlated lognormal density.

   (c) Write the partial differential equation one could solve to determine $E[\max(S_1(T), S_2(T))]$ with $S_1(0) = s_1$ and $S_2(0) = s_2$ and $\rho_{12} \neq 0$

4. Suppose $dS(t) = a(S(t),t)dt + \sigma(S(t),t)S(t)dB(t)$. Write a formula for $\int_0^T S(t)dS(t)$ that involves only Riemann integrals and evaluations.