Practice for the Final Exam.

The final exam is Thursday, May 3, from 5:10 to 7 pm in room 1302. You are allowed one piece of standard size paper with whatever you want written on it. Otherwise, the exam is closed book and closed notes. The exam will be graded to discourage guessing. A small amount of credit will be subtracted for each wrong answer. If you don't know the answer, your expected score will be higher (and the variance lower) if you leave it blank, which will receive zero credit. Points will be subtracted for any incorrect answer even if a correct answer also appears. Please cross out anything you think is wrong.

1. For each of the statements below, indicate whether it is true or false and explain why in a few words or sentences (but not more).

   (a) Let $Y(t)$ be an iid family of random variables with possible values $Y(t) = -1, 0, 1$, each with probability $1/3$. Let $X(t+1) = X(t) + Y(t)$, then $X(t)$ is a Markov chain whose state space is Z (the integers).

   (b) Let $Y(t)$ be as above and $X(t+1) = \max(X(t), X(t-1)) + Y(t)$, then $X(t)$ is a Markov chain whose state space is Z.

   (c) Let $W(t)$ be the pair of integers $W(t) = (X(t), X(t-1))$. If $X(t+1) = \max(X(t), X(t-1)) + Y(t)$ then $W(t)$ is a Markov chain whose state space is Z^2 (pairs of integers).

   (d) An single component exponential random variable is absolutely continuous with respect to a single component Gaussian.

   (e) A single component Gaussian is absolutely continuous with respect to an exponential.

   (f) Suppose $W(t)$ is a standard Brownian motion and $M(t) = \max_{s \leq t} W(s)$, then the Ito integral $Y(t) = \int_0^t M(s)dW(s)$ is well defined.

   (g) As above, $Y(t)$ is a martingale.

   (h) As above, $U(t) = \int_t^T M(s)dW(s)$ is a martingale for $t \leq T$.

   (i) Suppose $A(t) = \max_{t \leq s \leq t+1} W(s)$, then the Ito integral $Y(t) = \int_0^t A(t)dW(t)$ is well defined.

   (j) Let P be the measure corresponding to Brownian motion starting at $W(0) = 0$. Let Q be the measure corresponding to Brownian motion starting at $W(0) = 2$. The measures Q and P are completely singular with respect to each other.

2. Suppose $X(t)$ for $t \geq 0$ is the state at time t of a finite state space Markov chain. Suppose $u(k,t)$ is the probability that $X(t) = k$. Let $u(t)$ be the row vector whose components are the numbers $u(k,t)$. This satisfies the forward equation $u(t+1) = u(t)P$, where P is the transition matrix. Let $v(t)$ be a row vector of the same size that also satisfies $v(t+1) = v(t)P$, but with $v(k,t) < 0$ for some k and t. Which of the following is true:

(a) $v(k+1,t) \leq v(k,t)$ for all k and t.

(b) $\max_k v(k,t+1) \leq \max_k v(k,t)$

(c) $\sum_k v(k,t) = \sum_k v(k+1,t)$.

(d) $v(t)$ also satisfies $v(t+1) = Pv(t)$.

3. Suppose $X(t)$ is a Markov chain with a finite state space and transition matrix P. Suppose $X(0) = k_0$ is given and we want to estimate or evaluate $A = E[V(X(T))]$ ($T > 0$, $V(k)$ a given function). Which of the following is not a correct approach:

   (a) Create a row vector $u(0)$ with components $u(j,0) = 0$ if $j \neq k_0$ and $u(k_0,0) = 1$, then calculate $u(t+1) = u(t)P$ for $t = 0, 1, \ldots T-1$, then take $A = \sum_k u(k,T)V(k)$.

   (b) Create a column vector $f(T)$ with components $f(k,T) = 0$, then calculate $f(t) = Pf(t+1) + V$ for $t = T-1, \ldots, 0$, then take $A = f(k_0, 0)$. V is the column vector with components $V(k)$.

   (c) Create a column vector $f(T)$ with components $f(k,T) = V(k)$, then calculate $f(t) = Pf(t+1)$ for $t = T-1, \ldots, 0$, then take $A = f(k_0, 0)$.

   (d) Create a large number of independent sample paths $X(t,i)$, for $i = 1, \ldots, N$, each with transition probabilities P and each with $X(0,i) = k_0$, then take $A \approx \frac{1}{N}\sum_i V(X(T,i))$.

4. Let $X(t)$ be a simple one dimensional discrete time random walk whose possible moves are $X \to X+1$, $X \to X$ and $X \to X-1$. Let $M(X) = \max X(t), 0 \leq t \leq T$. Let $\mathcal{F}$ be the algebra generated by $M(X)$. Describe a typical atom in the partition that defines $\mathcal{F}$.

5. Let $N(t)$ be the number of people in a waiting room at time t. The room has *Poisson arrivals* with rate λ, which means that for small Δt, $\lambda\Delta t$ is roughly the probability of a new arrival. More technically, $P(N(t+\Delta t) = N(t)+1) = \lambda\Delta t + O(\Delta t^2)$. Each person in the room has a *departure rate* μ, which means that person leaves in the inverval $(t, t+\Delta t)$ with probability about $\mu\Delta t$, if he or she is present at time t. All choices for different people and different times are independent. (This process has a name that starts with "E", which I am whithholding).

   (a) Let m be the expected number of people in the room in steady state. Calculate m. Hint: Calculate $E[N(t+\Delta t) - N(t) \mid \mathcal{F}_t]$ and find the value of $E[N(t)]$ for which this is zero.

   (b) Calculate the steady state expected value of $N(t)^2$ and the steady state variance, σ^2.

   (c) Suppose λ is much larger than μ so that m is very large. How would you rescale $X(t) = N(t) - m$ so that it is approximately an Ornstein Uhlenbeck process? More precisely, find ξ and τ as functions of λ (and μ, which is fixed as $\lambda \to \infty$) so that $Y(s) = \xi(N(\tau s) - m)$ approximately satisfies $dY = -Y\,dt + dW$.

(d) Let u_n be the probability that $N(t) = n$ in the steady state. Find a formula relating u_n, u_{n-1}, and u_{n+1}. Show that the solution which is a probabilitiy distribution is $u_n = \dfrac{m^n e^{-m}}{n!}$. This is the Poisson distribution.

6. Let $Z(t)$ be a standard white noise process and define $X(t) = \int_{t-1}^{t} Z(s)ds$. Let $C(t_1, t_2) = \text{cov}(X(t_1), X(t_2))$.

   (a) Is $X(t)$ a Markov process?

   (b) Are $X(t_1)$ and $X(t_2)$ jointly normal?

   (c) What is the statistical relationship between $X(t_1)$ and $X(t_2)$ when $|t_2 - t_1| > 1$?

   (d) Evaluate $P(X(0) > 0 \mid X(.5) = 1)$. Express the answer using the cumulative normal distribution function $N(z) = P(Z \leq z)$, where Z is a standard normal random variable.

   (e) Show that the covariance function has the form $C(t_1, t_t) = f(t_2 - t_1)$, i.e., that it depends only on the time difference.

   (f) Calculate $f(s)$. Show that your answer is consistent with the result of part (c).

   (g) Suppose
   $$X(t) = \frac{1}{2\pi} \int e^{i\xi t} A(\xi) Z_2(\xi) d\xi \ .$$
   where $Z_2(\xi)$ is a white noise process related to $Z(t)$ in some way. Find a formula for $A(\xi)$.

   (h) Find a general formula for $\widehat{f}(\xi)$ in terms of A that does not depend on the particular form of A (from class).

   (i) Calculate $\widehat{f}$ from f in part (f) to see that this works.

7. Suppose $dX = aXdt + dW$ (X is simple Brownian motion with drift) and $X(0) = \frac{1}{2}$. Let τ be the hitting time $\tau = \min\{t \text{ with } X(t) = 0 \text{ or } X(t) = 1\}$.

   (a) Calculate $E[\tau]$.

   (b) Calculate $P(X(\tau) = 0)$.

8. Define $Y(t) = \int_0^t \sigma(s)dW(s)$, where $\sigma(t)$ is an adapted function of t.

   (a) Calculate $df(Y(t), t)$ in terms of derivatives of f, dW, and dt. Hint: There are at least two ways to do this. One is to calculate df in terms of dY and dt then calculate dY. Another is to figure it out more or less directly.

   (b) Assume $\sigma(t)$ and $w(t)$ are smooth functions of t, and write a formula for the solution of
   $$\dot{X} = \sigma(t) X \dot{W} \quad , \qquad X(0) = 1 \ .$$

   (c) Show how to correct the formula from part (b) to solve the SDE
   $$dX(t) = \sigma(t) X(t) dW(t) \quad , \qquad X(0) = 1 \ .$$
   You can find this solution in books under “random time change”.

9. Suppose $S(t)$ and $V(t)$ satisfy the SDE system

$$\begin{aligned} dS &= rSdt + \sqrt{V}SdW_1 \\ dV &= -\lambda\left(\sigma^2 - V\right)dt + \xi V dW_2 \end{aligned}$$

where W_1 and W_2 are standard but correlated Brownian motions with correlation coefficient ρ. The numbers r, λ, σ, ξ, and ρ are constants (parameters in the model). Describe the PDE problem we could solve to evaluate

$$f(s, v, t) = E_{x,v,t}\left[|S(T) - K|\right] \ .$$

Give any initial, final, or boundary conditions needed.