Assignment 10.

Due April 5.

1. Consider an ordinary differential equation

$$dx = f(x,t)dt \; . \tag{1}$$

We make an approximate solution using a time step Δt, discrete times $t_k = k\Delta t$, and an approximate solution $x_k \approx x(t_k)$. One approximation method is *forward Euler*. This is an approximation using (1) with $dt = \Delta t$: $x(t+\Delta t) - x(t) \approx f(x(t),t)\Delta t$. The discrete method based on this is

$$x_{k+1} - x_k = f(x_k, t_k)\Delta t \; . \tag{2}$$

Another approximation method uses (1) with $dt = -\Delta t$ to get $x(t - \Delta t) - x(t) \approx -f(x(t),t)\Delta t$. We apply this with $t = t_{k+1}$ to get

$$x_{k+1} = x_k + f(x_{x+1}, t_{k+1})\Delta t \; . \tag{3}$$

This is the *backward Euler* method. To use it, we have to solve the equation $x_{k_1} - f(x_{k+1}, t_{k+1})\Delta t = x_k$ for x_{k+1}. Sometimes it is easy to do this. From a mathematical point of view, the methods seem be be based on the same kind of approximation and therefore would have a comparable accuracy. It is not hard to show that if $\Delta t \to 0$ and $n \to \infty$ with $n\Delta t = T$ fixed, then either (2) or (3) leads to $x_n \to x(T)$. This exercise explores what happens when we apply these two methods to the stochastic differential equation

$$dX = \sigma X dW \quad , \quad X(0) = 1 \; . \tag{4}$$

The forward and backward Euler approximations are

$$X_{k+1} = X_k + \sigma X_k \Delta W_k \; , \tag{5}$$

and

$$X_{k+1} = X_k + \sigma X_{k+1} \Delta W_k \; , \tag{6}$$

where $\Delta W_k = W(t_{k+1}) - W(t_k)$.

(a) Show that

$$1 + \epsilon = \exp(\epsilon - \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 - \frac{1}{4}\epsilon^4 + O(\epsilon^5)) \tag{7}$$

Hint: Use the Taylor series for log.

(b) Derive a formula of the type

$$X_n = \exp\left(\sum_{k=0}^{n-1} \Delta W_k - \frac{1}{2}\sum_{k=0}^{n-1} + *** - *** + \sum_{k=0}^{n-1} O\left(|\Delta W_k|^5\right)\right)$$

(c) Neglecting the $O\left(|\Delta W_k|^5\right)$ term, calculate the limit as $\Delta t \to 0$ and $n \to \infty$ with $n\Delta t = T$ fixed of each of the first four sums in the exponent.

(d) Show that this is the solution of (4) we had in class.

(e) Repeat steps (b), (c), and (d) for the backward Euler method (6) to see what X_n converges to.

2. We know that $\sqrt{t}$ is the order of magnitude of $W(t)$ in general. One manifestation of this is that the distribution of $W(t)/\sqrt{t}$ is independent of t (in particular, it's standard normal for any t). It should not be surprising that it is possible to show that $(W(t)/\sqrt{t})/\sqrt{t} = W(t)/t \to 0$ as $t \to \infty$. Use this to show that the solution of (4) satisfies $X(t) \to 0$ as $t \to \infty$. Remember that $X(t)$ is a martingale and $E[X(t)] = 1$ for all t.

3. Suppose $W(t)$ is a Brownian motion and $f(W(t), t)$ is a martingale. Show that f satisfies the backward heat equation. Use this to show that if $f(x, T) = V(x)$, then $f(x, t) = E_{x,t}[V(W(T))]$. Find $f(x, t)$ when $V(x) = e^{i\xi x}$. What does this say about the solution to this backward equation with oscillatory final values as we move away from the final time? How does this behavior depend on the frequency of oscillation?

4. Here is a similar approach to another backward equation in the notes, but different from the approach there.

   (a) Let $Y(t) = \int_t^T V(W(s))ds$. Show that $dY = -V(W(t))dt$ in the sense of Ito.

   (b) What PDE does $f(x, t)$ have to satisfy in order for $f(W(t), t) + Y(t)$ to be a martingale?

   (c) What PDE does

   $$f(x, t) = E_{x,t}\left[\int_t^T V(W(s))ds\right]$$

   satisfy? What are its final conditions?

   (d) Find the solution to the PDE with final conditions when $V(x) = x^2$. Hint: For each t, the solution is a polynomial of degree 2.

   (e) Calculate $E_{x,t}\left[\int_t^T W(s)^2 ds\right]$ directly using properties of Brownian motion. You should get the same answer.