Assignment 12. Due April 19. Corrections April 18: 1(d) and (e) p_k changed to p_n, clarification on the double role of p in question 4, (2a) corrected $e^{-T/\tau}$ became $e^{-t/\tau}$, (3) R_n replaced by X_n

1. Let S_n be a sequence of random variables. There are several ways to interpret the statement $S_n \to 0$ as $n \to \infty$. Convergence *in probability*, written $S_n \xrightarrow{P} 0$, is the statement that for an $\epsilon > 0$ $P(|S_n| > \epsilon) \to 0$ as $n \to \infty$. If you are planning to choose a single large n and are hoping that S_n is close to zero, this int the kind of convergence you need. A stronger kind of convergence is *almost sure* convergence. We say $S_n \to 0$ almost surely (abbreviated a.s.) if $P(S_n \to 0 \text{ as } n \to \infty) = 1$. You might be interested in this kind of convergence if you are going to use S_n for many different values of n and you need all of them to be small. This exercise and the next illustrate that it is easier to know a single S_n is close to zero than to know many of them are.

   Let X_k be independent Bernoulli random variables with $p_k = P(X_k = 1)$ and $1 - p_k = P(X_k = 0)$. Let M be the number of k with $X_k = 1$. If $M < \infty$, let N be the largest k with $X_k = 1$. Take $N = \infty$ if $M = \infty$.

   (a) Show that if $p_k > 0$ is a sequence of numbers with $\sum_{k=1}^{\infty} p_k = \infty$, then for any $N > 0$, $\sum_{k=N}^{\infty} p_k = \infty$.

   (b) Show that $X_n \xrightarrow{P} 0$ as $n \to \infty$ if and only if $p_n \to 0$ as $n \to \infty$.

   (c) Show that $M < \infty$ is equivalent to $X_n \to 0$ as $n \to \infty$. Hint: the only way for a Bernoulli to be close to zero it to be equal to zero.

   (d) Show that if $\sum_{n>0} p_n < \infty$, then $X_n \to 0$ as $n \to \infty$ a.s. This is the actual Borel Cantelli lemma. Hint: $E(M) = \sum_{n>0} p_n < \infty$.

   (e) Show that if $\sum_{n>0} p_n = \infty$, then the X_n do not converge to 0 as $n \to \infty$ almost surely. This sometimes is called the converse of the Borel Cantelli lemma, but that is not strictly correct since the actual lemma does not require the X_n to be independent while this does. Hint: Let A_N be the event that $X_k = 0$ for all $k \geq N$. If ω is an outcome with $X_n \to 0$ as $n \to \infty$, then $\omega \in A_N$ for some N (why?). If $P(A_N) = 0$ for all N then $P(\cup_N A_N) = 0$ (why?), so $P(X_n \to 0) = 0$. For any $N > 0$, $P(X_N = 0) = (1 - p_N)$, $P(X_N = 0 \text{ and } X_{N+1} = 0) = (1 - p_N)(1 - p_{N+1})$, etc. The probability that there are no $X_k = 1$ for $N \leq k \leq N + n$ is (show $1 - t \leq e^{-t}$ if $t \geq 0$)

$$\prod_{k=N}^{N+n} (1 - p_k) < \exp\left(- \sum_{k=N}^{N+n} p_k \right) \to 0 \ ,$$

as $n \to \infty$.

2. Let T_n be independent exponential random variables with $\tau_n = E[T_n]$.
   (a) Show that $P(T_n > t) = e^{-t/\tau_n}$.
   (b) Show that $T_n \xrightarrow{P} 0$ as $n \to \infty$ if and only if $\tau_n \to 0$ as $n \to \infty$. Hint: use part (a).
   (c) Find a specific sequence τ_n so that $T_n \xrightarrow{P} 0$ as $n \to \infty$ but, with probability 1, $\lim\limits_{n\to\infty} T_n$ does not exist. Hint: Use part (a) and the reasoning of question 1, part (e).

3. One advantage of the weak solution point of view is that it makes it possible to do things like this. Suppose $X_1, \ldots, X_n$ are n independent standard Brownian motions and
$$R(t) = \left(X_1^2 + \cdots + X_n^2\right)^{1/2} .$$
   (a) Compute dR using Ito's lemma, the version for $f(W(t), t)$, where $W(t)$ is an n component standard Brownian motion.
   (b) Use the result of part (a) to calculate $a(r)$ and $b(r)$ so that
$$\begin{aligned} E\left[R(t+\Delta t) - R(t) \mid \mathcal{F}_t\right] &= a(R(t))\Delta t + O\left(\Delta t^2\right) \\ E\left[(R(t+\Delta t) - R(t))^2 \mid \mathcal{F}_t\right] &= b(R(t))^2\Delta t + O\left(\Delta t^2\right) \end{aligned}$$
   (c) Use the result to express $R(t)$ as the weak solution of the SDE $dR = a(R)dt + b(R)dW$.

4. Another advantage of the weak solution point of view is that you can make diffusion approximations to discrete processes by simple scalings and a little algebra. Consider the urn process for large n and fixed p. Let $X(k)$ be the number of green balls in the urn after k steps. We want to define a diffusion process, $Y(t)$ that is an approximation to the X process. This involves *scalings*, changing the X and time scales by powers of n so that the rescales process, $Y(t)$, has order one changes in order one time. Let $Y_n(t)$ be $X(k)$ rescaled. The convergence theorem (which we don't prove) says that the probability measure in path space for $Y_n(t)$ converges in the *weak sense* (which we do not define) to the probability measure for $Y(t)$.

   We have to rescale in in x and t. That is, we choose $Y_n = C_X n^{-p}(X - \mu)$ and $\Delta t = C_t n^q$ and then the stochastic process $Y_n(k\Delta t) = C_X n^{-p}(X(k) - \mu)$. The X processes are very different from each other as $n \to \infty$, but simply rescaling to $Y_n(t)$ gives processes that are more and more alike. *Added April 18:* The scaling exponent p used in defining Y_n has nothing to do with the p used in defining the urn model. It may help to change the definition to $Y_n = C_X n^{-r}(X - \mu)$ then change part (b) below to: "Choose r and C_X above ...". Bear this in mind when doing question 5 below.

   (a) Define and evaluate $\mu = \lim_{k\to\infty} E[X(k)]$ and $\sigma^2 = \lim_{k\to\infty} \mathrm{var}[(X(k)]$. Hint: look in the notes.

(b) Choose p and C_X above so that $\lim_{t\to\infty} \text{var}[Y_n(t)] = 1$.

(c) Choose q and C_t so that

$$E\left[Y_n(t+\Delta t)^2 \mid Y(t) = 0\right] = \Delta t \ .$$

(d) Find simple expressions for $a(y)$ and $b(y)$ so that

$$\begin{aligned} E\left[Y_n(t+\Delta t) - Y_n(t) \mid \mathcal{F}_t\right] &\approx a(Y_n)\Delta t + O\left(\Delta t^2\right) \\ E\left[\left(Y_n(t+\Delta t) - Y_n(t)\right)^2 \mid \mathcal{F}_t\right] &\approx b(Y_n)^2\Delta t + O\left(\Delta t^2\right) \end{aligned}$$

Where the approximations become accurate in the limit $n \to \infty$ with Y fixed. This is the Ornstein Uhlenbeck approximation to the urn process.

5. This exercise asks you to simulate the urn process with various values of n and p to see that they look nearly the same when rescaled. You should have a program that simulates the urn process that takes n and p as parameters. Start with $X(0) = np$, which is the mean value. Continue the path until the first k with $|X(k) - np| > \sigma$, where $\sigma = \sqrt{np(1-p)}$ is the standard deviation of the steady state probability distribution. Output that k, which is the *hitting time*. The simplest way to keep track of the k values is to have an array called something like `kCount` of size, say, $kMax$. Start by setting `kCount[j] = 0` for all j (this will be for $j = 0, \ldots, kMax - 1$ or $j = 1, \ldots, kMax$ depending on the system you are using.) Every time you simulate a path and get a random exit time (or hitting time), k, record that time using `kCount[k] = kCount[k] + 1` (or `kCount[k]++;` in C/C++, or `kCount(k) = kCount(k) + 1;` in Matlab). To keep your program from crashing, you should output $k = kMax$ (or $k = kMax - 1$) if the path has not hit the boundary by then. You should choose $kMax$ so that this is rare. Do a run that generates L independent paths and record the L hitting times. Then create an estimate of the cumulative distribution function $F(j) = P(k \leq j)$ as

$$F(j) \approx \sum_{k \leq j} kCount(k)/L \ .$$

If L is large enough, this will be a reasonably smooth curve. Now choose a p (not too close to 0 or 1) and run the program for various values of n (not too small) to see that if you plot $F(j)$ with time rescaled as in question (4), then curves for different values of n are nearly the same. Do they agree for different p values as well?