Assignment 13. Due April 26.
Corrections April 24: in 3(c) the integral is dx, not dt. April 26: a formula in 1b corrected.

1. Let

$$Y(t) = \rho\left(X_1^2 + \cdots + X_n^2\right) , \qquad (1)$$

where the $X_k(t)$ are independent Brownian motions with variance t but $X_k(0) \neq 0$. Let $u(y,t)$ be the probability density for $Y(t)$.

(a) Find the value of ρ so that $dY = \sqrt{Y}dW + adt$. Here a is a constant that you need to identify.

(b) Create a histogram that gives the approximate shape of $u(y,1)$ using the right side of (1). Choose a large number, L, say $L = 10^5$ or $L = 10^6$, depending on your computer and your patience. Choose a small Δy, say $\Delta y = .05$ or $\Delta y = .01$. Define the bin B_j to be the interval $B_j = [j\Delta y, (j+1)\Delta y)$. For each i in the range $1 \leq i \leq L$, generate n independent standard normals $(X_{i,1}, \ldots, X_{i,n})$ and set

$$Y_i = \rho\Big((X_{i,1} + c)^2 + \cdots + X_{i,n}^2\Big) ,$$

with $c = 1/\sqrt{\rho}$ so that $Y_i = 1$ if all the $X_{i,k}$ are zero. We use $X_1 + c$ instead of X_1 in order to have a normal with mean c and variance one, which means that $Y(0) = 1$. You need not store the *samples* Y_i, only the *bin counts* $H_j = \#\{i \text{ with } Y_i \in B_j\}$. Let $y_j = (j + \frac{1}{2})\Delta y$ be the center of bin B_j. The histogram estimate of the probability density is

$$u(y_j, 1) \approx \widehat{u}(y_j, 1) = \frac{1}{L\Delta y} H_j . \qquad (2)$$

Your program will have to have a maximum y_j, but it should be very rare to have a sample larger than about 7. Write your program so that it will not crash if that should happen.

(c) Write a program to create simulated sample paths from the SDE of part (a) starting with initial value $Y(0) = 1$. Estimate the probability density for the random variable $Y(1)$ using the histogram method from part (b). This requires you to generate L independent sample paths up to time $t = 1$ and put the results in bins. Generate each sample path using the *forward Euler* method. Choose a small *time step*, Δt, say $\Delta t = .1$, and implement the approximate relation $\Delta y \approx a\Delta t + \sqrt{Y}\Delta W$ by

$$Y_{m+1} = Y_m + a\Delta t + \sqrt{Y_m}\sqrt{\Delta t} Z_m , \qquad (3)$$

where the Z_m are independent standard normals. Explain why the distribution of $\sqrt{\Delta t} Z_m$ is the same as the distribution of $\Delta W = W(t_{m+1}) - W(t_m)$. Here $t_m = m\Delta t$ and Y_m is the stochastic approximation to $Y(t_m)$. Take $Y_0 = 1$. The estimate of $Y(1)$ is Y_m with $t_m = 1$. Plot the estimates of the probability density make by simulating sample paths and the estimates from part (b) on the same graph to compare.

There are two sources of error, statistical and inexact sample paths. The statistical error is reduced by increasing L. The sample paths are made more accurate by reducing Δt. The harder you push your computer, the better will be the agreement between the two curves.

(d) Show that the approximate time step formula (3) has the properties that

$$
\begin{aligned}
E[\Delta Y \mid \mathcal{F}_m] &= a\Delta t \\
E[\Delta Y^2 \mid \mathcal{F}_m] &= Y_m \Delta t \quad \text{(as it is supposed to be)} \\
E[\Delta Y^4 \mid \mathcal{F}_m] &= O\left(\Delta t^2\right) .
\end{aligned}
$$

This is the reason the distribution of the approximate sample paths (3) converges to the distribution of sample paths of the SDE as $\Delta t \to 0$.

2. An *Ornstein Uhlenbeck* process is an adapted process $X(t)$ that satisfies the Ito differential equation

$$
dX(t) = -\gamma X(t) dt + \sigma dW(t) . \tag{4}
$$

We have to be careful in applying Ito's lemma because $X(t)$ is not simply a function of $W(t)$ and t, but it also depends on values of $W(s)$ for $s < t$.

(a) Examine the definition of the Ito integral and verify that if $g(t)$ is a non random differentiable function of t, and $dX(t) = a(t)dt + b(t)dW(t)$, with a random but bounded and nonanticipating $b(t)$, then $d(g(t)X(t)) = \dot{g}(t)X(t)dt + g(t)dX(t)$. It may be helpful to use the Ito isometry formula (paragraph 1.17 of lecture 7).

(b) Bring the drift term to the left side of (4), multiply by $e^{\gamma t}$ and integrate (using part a) to get

$$
X(T) = e^{-\gamma T} X(0) + \sigma \int_0^T e^{-\gamma(T-t)} dW(t) . \tag{5}
$$

(c) Conclude that if $X(0)$ is Gaussian or deterministic, then $X(T)$ is Gaussian for any T.

(d) Use the SDE (4) to find ODEs for $\mu(t) = E[X(t)]$ and $v(t) = E[X(t)^2]$. Hint: $\frac{d}{dt} E[f(X(t),t)]dt = E[f_x dX + f_t dt + \frac{1}{2} f_{xx} (dX)^2] = E[f_x a + f_t + \frac{1}{2} f_{xx} b^2]dt$. Apply this with $f = x$ and $f = x^2$.

(e) Use the result of part (d) to show that the probability density for $X(T)$ has a limit as $T \to \infty$. Find the limit by computing the mean and variance directly from the integral (5) using the Ito isometry formula.

(f) Suppose $X(t)$ is defined and satisfies (4) for $t < 0$ and is bounded or slowly growing as $t \to -\infty$. Suppose that $W(t)$ is defined for $t < 0$. Use the reasoning that led to (5) to derive a formula for $X(T)$ as a single integral involving Brownian motion for over all $t \leq T$. Show that the formula is identical to the one from Assignment 8, question 3.

(g) Contrast the large time behavior of the Ornstein Uhlembeck process with that of Brownian motion.

3. In this exercise and the next (next week), we will calculate some solutions of the backward and forward equations for the Ornstein Uhlenbeck process. We will be able to check directly that the soltuion to the backward equation is an expected value, the solution to the forward equation is a probability density, and that the duality relations hold.

   (a) Write the backward equation for $f(x,t) = E_{x,t}[V(X(T)]$, when $X(t)$ satisfies (4).

   (b) Show that the backward equation has (Gaussian) solutions of the form $f(x,t) = A(t)\exp(-s(t)(x-\xi(t))^2/2)$. Find the differential equations for A, ξ, and s that make this work. This is the *ansatz* method for solving a PDE. You guess the general form of the solution and see what the parameters have to do to make it work.

   (c) Show that $f(x,t)$ does not represent a probability distribution, possibly by showing that $\int_{-\infty}^{\infty} f(x,t)dx$ is not a constant.

   (d) What is the large time behavior of $A(t)$ and $s(t)$? What does this say about the nature of an Ornstein Uhlenbeck reward that is paid long in the future as a function of starting position?