Assignment 3.

Given January 25, due February 8.
Objective: Markov chains, II and lattices.

Revised February 2.

1. We have a three state Markov chain with transition matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \; .$$

Some of the transition probabilities are $P(1 \to 1) = \frac{1}{2}$, $P(3 \to 1) = \frac{1}{3}$, and $P(1 \to 2) = \frac{1}{4}$. Let $\tau = \min(t \mid X_t = 3)$. Suppose that at time $t = 0$, all states are equally likely.

   (a) Consider the quantities $u(j,t) = P(X(t) = j \text{ and } \tau > t)$. Find a matrix evolution equation for a two component vector made from the $u(j,t)$ and a submatrix, $\widetilde{P}$, of P.

   (b) Solve this equation using the the eigenvectors and eigenvalues of $\widetilde{P}$ to find a formula for $m(t) = P(\tau = t)$.

   (c) Use the answer of part (b) to find $E[\tau]$. It might be helpful to use the formula

$$\sum_{t=1}^{\infty} tP(\tau = t) = \sum_{t=1}^{\infty} P(\tau \geq t) \; .$$

   Verify the formula if you use it.

   (d) Consider the quantities $f(j,t) = P(\tau \geq t \mid X(0) = j)$. Find a matrix recurrence for them.

   (e) Use the matrix method to find a formula for $f(j,t)$.

2. This problem explores a Markov chain observed at random times, and reviews some linear algebra in the process.

   (a) Suppose A is an $n \times n$ matrix with $\|A\| < 1$ (Any matrix norm will do.). We write A^t for A to the power t, not the transpose of A. If $t = 0$, then $A^t = I$, the identity matrix. Show that

$$\sum_{t=0}^{\infty} A^t = (I - A)^{-1} \; .$$

   Hint: this is the same as

$$(I - A)\left(I + A + A^2 + \cdots\right) = I \; .$$

(b) Let P be the transition matrix for a Markov chain. Let $f = (f_1, \ldots, f_n)^*$ be an n component column vector (writing f^* for the transpose of f). The max norm, or L^∞ norm, of f is

$$\|f\|_{L^\infty} = \max_k |f_k| \ .$$

Show that if $g = Pf$, then $\|g\|_{L^\infty} \leq \|f\|_{L^\infty}$. This is essentially the maximum principle we did in class. Show that if $f = \mathbf{1}$ (the vector with all components equal to one), then $\|g\|_{L^\infty} = \|f\|_{L^\infty}$ This implies that $\|A\|_{L^\infty} = 1$.

(c) Suppose P_1 and P_2 are two $n \times n$ transition matrices. Suppose we toss a coin that gives H with probability r and use P_1 if H and P_2 otherwise. Show that the resulting transition matrix is $rP_1 + (1-r)P_2$.

(d) Suppose τ (the Greek letter "tau") is a geometric random variable with parameter r. That means that τ is a non-negative integer with $P(\tau = 0) = r$, $P(\tau = 1) = (1-r)r$, $P(\tau = 2) = (1-r)^2 r$, etc. We get τ by tossing a coin (independent tosses) until the first H. Show that τ has the property that, for all $t \geq 0$, $P(\tau = t \mid \tau \geq t) = P(\tau = 0)$. We interpret this by thinking of τ as the time something breaks. If it has not broken before time t, it as good as new.

(e) Suppose we run a Markov chain starting with state $X_0^{(0)} = Y_0$ using transition matrix P and let $Y_1 = X_\tau^{(0)}$, then run the P Markov chain again with $X_0^{(1)} = Y_1$ and let $Y_2 = X_\tau^{(1)}$ (independent of $X^{(0)}$), and in this way create a path $Y = (Y_0, Y_1, \ldots)$. Show that Y is a Markov chain and find a formula for its transition matrix in terms of P and r. Hint: use parts (a), (c), and (d).

(f) Let

$$P = \begin{pmatrix} .8 & .2 \\ .2 & .8 \end{pmatrix}$$

be the transition matrix for a two state Markov chain. Suppose $X_0 = 1$. Find a simple explicit formula for $u(1,t) = P(X_t = 1)$. Hint compute the first few by hand until you see the general pattern.

(g) Combine the formulas from part (f) and part (d) to get a formula for $P(X_\tau = 1 \mid X_0 = 1)$, assuming $r = \frac{1}{2}$.

(h) Do the matrix inversion of parts (a) and (e) to recompute the result of part (g). The answers should be the same.

3. This exercise reviews more linear algebra and explains how to create a martingale that is useful for proving the Central Limit Theorem for Markov Chains. The *rank* of a matrix is the dimension of the vector space spanned by its columns. If A is an $n \times n$ matrix, the *row kernel* of A, K_r, (also called the left kernel, or the kernel of A^*) is the vector space of row vectors u so that $uA = 0$. If A has rank $n - r$, if K_r has dimension r. The column kernel (or simply the kernel) of A, K_c, is the vector space of column vectors, f so that $Af = 0$. A theorem of linear algebra says that the dimensions of K_c and K_r are equal. If K_r is nontrivial, then there are vectors g so that there is no solution to the equations $Af = g$. If $u \in K_r$, then $uAf = ug$. Since the left side is

zero, the right side also must be zero. A theorem of linear algebra says that $Af = g$ has a solution if $ug = 0$ for all $u \in K_r$, and that $uA = v$ has a solution if $vf = 0$ for all $f \in K_c$. If A, has rank $n - 1$, then K_r and K_c are one dimensional. That means that there is a unique row vector, u, so that $uA = 0$, unique in the sense that if v is another row vector with $vA = 0$ then v is a multiple of u. If

(a) Let P be an $n \times n$ matrix and λ a real or complex number. Use the above discussion (i.e., not determinants) to show that there is a non-zero row eigenvector with $uA = \lambda u$ if and only if there is a non-zero column eigenvector with $Af = \lambda f$. Hint: take $A = P - \lambda I$.

(b) Let P be the transition matrix of a Markov chain and $\mathbf{1}$ the column vector with all entries equal to one. Show that $P\mathbf{1} = \mathbf{1}$. If the Markov chain is *nondegenerate* (definition given later), then $P - I$ has rank $n - 1$. Show that in this case, there is a unique row vector, π, with $\pi P = \pi$ and $\sum_{k=1}^{n} \pi_k = \pi\mathbf{1} = 1$ (the right side being the number one). Hint: You may assume that if π has $\pi P = \pi$ then all the components of π have the same sign. This π is a probability distribution on the state space. Show that if $P(X_t = j) = \pi_j$, then $P(X_{t+1} = j) = \pi_j$, i.e. that π a *steady state* probability distribution for the Markov chain.

(c) Let f be a function defined on the state space of a Markov chain, and let

$$S_t = \sum_{t=0}^{t} f(X_s) \; .$$

Show that if $E_\pi[f(X)] = \sum_j \pi_k f(j) = 0$, then there is a function g, defined on the state space, so that $M_t = S_t - g(X_t)$ is a martingale. The definition of a martingale is that $E[M_{t+1} \mid \mathcal{F}_t] = M_t$. In this case, $M_{t+1} = M_t + f(X_{t+1}) - (g(X_{t+1}) - g(X_t))$, so the martingale condition is that

$$E\left[f(X_{t+1}) \mid X_t = j\right] = E\left[g(X_{t+1}) - g(X_t) \mid X_t = j\right] \quad \text{for all } j.$$

Formulate this as a system of equations for the unknown column vector, g.