Assignment 5.

February 22.
Corrected version posted Feb 21.

1. In the urn process, suppose balls are either *stale* or *fresh*. Assume that the process starts with all n stale balls and that all replacement balls are fresh. Let τ be the first time all balls are fresh. Let $Y(t)$ be the number of stale balls at time t. Show that $Y(t)$ is a Markov chain and write the transition probabilities. Use the backward or forward equation approach to calculate the quantities $a(y) = E_y(\tau)$. This is a two term recurrence relation ($a(y+1) = \text{something} + a(y)$) that is easy to solve. Show that at time τ, the colors of the balls are iid red with probability p. Use this to explain the binomial formula for the invariant distribution of colors.

2. For the urn process, define $f(k, m, t) = P(X(T) = m \mid X(t) = k$ (defined for $t \leq T$).

   (a) For a fixed constant m, find the *final values* $f(k, m, T)$ and the backward equation that computes the numbers $f(k, m, t)$ from the numbers $f(k, m, t+1)$. Write the backward equation explicitly using formulas for the transition probabilities.

   (b) Check by direct computation that if $g(k, t)$ is any solution to the backward equation with $g(k, t+1) = g(k, t)$ for all k, then g is a constant (independent of k as well as independent of t). Hint: start by showing that $g(1, t) = g(0, t)$, then $g(2, t) = g(1, t)$, etc.

   (c) Write a program to calculate the f from part (a) with $T = 100$, $n = 30$, and $p = .5$. The last two are as in figure 2 of lecture 2. Plot the solution at a number of times (e.g. $t = 90, 80, 70, 50, 30, 0$) What does the solution say about how $P(X(T) = 15$ depends on $X(0)$ when T is large? How do you explain this?

   (d) (Only if you are curious and time permitting, worth 0 points) Write a program that simulates the urn process, start it many times from specific initial locations and times and verify the results of part (c) by direct simulation.