Assignment 8.

Due March 22.
Corrected March 21.

1. Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Find a formula for $E[e^X]$. Use this to find $E[e^{aX}]$ as a function of a, μ, and σ.

2. Suppose $X = (X_1, X_2, X_3)$ is a 3 dimensional Gaussian random variable with mean zero and covariance
$$E[XX^*] = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} .$$
Set $Y = X_1 + X_2 - X_3$ and $Z = 2X_1 - X_2$.
   (a) Write a formula for the probability density of Y.
   (b) Write a formula for the joint probability density for (Y, Z).
   (c) Find a linear combination $W = aY + bZ$ that is independent of X_1.

3. The Ornstein Uhlenbeck process is one of the best examples in stochastic calculus, both because it has many practical applications and because it is a simple situation in which we can calculate everything. Let $Z(t)$ be a standard white noise, and $\lambda > 0$ a positive decay rate. Define
$$X(t) = \int_{-\infty}^{t} e^{-\lambda(t-t_1)} Z(t_1) dt_1 . \tag{1}$$
   (a) Show that $X(t)$ informally is the solution to the differential equation
$$\frac{dX}{dt} = -\lambda X + Z(t) , \tag{2}$$
and that the formula (1) gives the solution to this equation. This is informal because $X(t)$ as given by (1) is not differentiable and $Z(t)$ is not an honest function. The Ito calculus gives a more formal way to express the same thing.
   (b) Use the fact that $Z(t)$ is stationary to show that $X(t)$ is stationary. This means that the statistical properties of $\widetilde{Z}(t) = Z(t - t_0)$ are the same as the statistical properties of $Z(t)$, and the same for $X(t)$. Hint: change variables in the integral (1).
   (c) Calculate the correlation function
$$R(s) = E\left[X(t)X(t+s)\right] .$$

Hint: First show that $R(s) = R(-s) = E[X(0)X(s)]$, using the fact that X is stationary. Then suppose $s > 0$ and write $X(s)$ as an integral with respect to another variable, t_2, reverse the order of integration and use the correlation formula for white noise. Note that if $R(s) = e^{-s}$ for $s > 0$, then $R(s) = e^{-|s|}$ for all s.

(d) Calculate $\widehat{R}(\xi)$, the Fourier transform of the correlation function. Hint: break the integral over s into a part with $s > 0$ and a part with $s < 0$.

(e) Find a representation of $\widehat{X}(\xi)$ in terms of $\widehat{Z}(\xi)$, the Fourier transform of white noise $\widehat{X}(\xi) = A(\xi)\widehat{Z}(\xi)$. Hint: There are two ways to do this (at least). One applies the Fourier transform directly to (1) and reverses the order of integration. The other substitutes the Fourier representation of Z into (1), reverses the order of integration and recognizes the result as a Fourier representation of $X(t)$.

(f) Show that $|A(\xi)|^2 = \widehat{R}(\xi)$, as theory says it should.

(g) Let $\mathcal{F}_t$ be the $\sigma-$algebra generated by the values[1] $X(s)$ for $s \leq t$. Find a formula for $E[X(t+s) \mid \mathcal{F}_t]$. Hint: Use the fact that disjoint parts of $Z(t)$ are independent mean zero. Use this formula to give another derivation of the formula for R in part (c).

(h) The differential equation (2) suggests that $X(t)$ is a Markov process. This means that the future depends on the past only through the present. In this case, that follows from the fact that there is a formula for $X(t+s)$ in terms of $X(t)$ and random variables independent of $X(t_1)$ for $t_1 \leq t$. Give the formula (or find it in part (g)) and explain how it implies the Markov property.

[1] This is the same as the algebra generated by the "values" of $Z(s)$ for $s \leq t$.