Sample questions for the final exam

Instructions for the final:

- The final is Monday, December 19 from 7:10 to 9pm
- Explain all answers, possibly briefly. A correct answer with no explanation may receive no credit.
- You will get 20% credit for not answering a question. Points will be subtracted if you give a wrong answer.
- Cross off anything you think is wrong. You will have points subtracted for wrong answers even if the correct answer also appears.
- You may use one $8\frac{1}{2} \times 11$ piece of paper, a *cheat sheet* with anything you want written on it.
- The actual exam will be shorter than this.

True/False. In each case, state whether the statement is true of false and give a few words or sentence of explanation.

1. Suppose $Z = (Z_1, \ldots, Z_n)^2$ with the $Z_k \sim \mathcal{N}(0,1)$ independent. Suppose that A and B are $n \times n$ non-singular matrices with $A \neq B$. Then $X = AZ$ and $Y = BZ$ are not equal (almost surely).
2. Suppose $Z = (Z_1, \ldots, Z_n)^2$ with the $Z_k \sim \mathcal{N}(0,1)$ independent. Suppose that A and B are $n \times n$ non-singular matrices with $A \neq B$. Then $X = AZ$ and $Y = BZ$ have different probability distributions.
3. Suppose $\mathcal{F}_t$ is a filtration on a probability space Ω. Suppose X is a random variable and $X_t = E[X \mid \mathcal{F}_t]$. Then X_t is a Markov chain.
4. If X_t is a martingale, then X_t is a Markov chain. (Give a counter-example if it's false.)
5. If $dX_t = \gamma_t X_t dt + dW_t$ and $dY_t = \lambda Y_t dt + dX_t$, then Y_t is an Ornstein Uhlenbeck process.
6. If S_t is a geometric Brownian motion then S_t^3 is a geometric Brownian motion.

Multiple choice In each case only one of the possible answers is correct. Identify the correct answer and explain why.

7. Suppose X_t is a Markov process, and $t_1 < t_2 < T$. Suppose the $\sigma-$algebra $\mathcal{F}_t$ is generated by X_s for $s \leq t$ and the $\sigma-$algebra $\mathcal{G}_t$ is generated by X_t only. Which of the following is a consequence of the *tower property*

   (a) $\mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2}$.
   (b) There is a function $f(x,t)$ so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t, t)$.
   (c) $f(x,t_1) = E_{x,t_1}[f(X_{t_2}, t_2)]$, where $f(x,t)$ is the function so that $E[V(X_T) \mid \mathcal{F}_t] = f(X_t,t)$.
   (d) $E[V(X_T) \mid \mathcal{F}_t] = E[V(X_T) \mid \mathcal{G}_t]$

8. Suppose X_t and Y_t are geometric Brownian motions that satisfy $dX_t = \mu_X X_t dt + \sigma_X X_t dW_{1,t}$, and $dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_{2,t}$. Here $W_{1,t}$ and $W_{2,t}$ are independent Brownian motions, $\mu_X \neq \mu_Y$ and $\sigma_X \neq \sigma_Y$. Let $S_t = X_t + Y_t$. Then

   (a) S_t is a Markov process but not a diffusion process.
   (b) S_t is a geometric Brownian motion.
   (c) S_t is a diffusion and a Markov process.
   (d) S_t is a diffusion process but not a Markov process.

General questions.

9. Suppose X_t satisfies the SDE $dX_t = adt + dW_t$ (a is a constant) with $X_0 = 0$. What is the distribution of X_3 given that $X_1 = -1$, $X_4 = 2$, and $X_5 = 1$? Give the type of distribution and the parameters that describe the conditional distribution of X_3 completely.

10. If X is an exponential random variable with $m = E[X]$, then $E[X^3] = Cm^3$, where C does not depend on m. Find a formula for C.

11. Suppose $X = \int_0^1 t dW_t$ and $Y = \int_0^1 t^2 dW_t$. Calculate $E[X^2]$ and $\text{cov}(X,Y)$.

12. Suppose $dX_t = a(X_t)dt + b(X_t)dW_t$. Suppose $s = t^2$ and $Y_s = X_{t^2}$. Find the SDE that Y_s satisfies in the weak sense. Hint: calculate $E[dY \mid \mathcal{F}_s]$, where $dY = Y_{s+ds} - Y_s$, etc.

13. Suppose X_t is an Ornstein Uhlenbeck process $dX_t = -\gamma X_t dt + \sigma dW_t$. Calculate $d\cos(X_t)$ using the Ito calculus.

14. Give an example of a continuous time Markov process whose sample paths are not continuous functions of time.

15. Consider the limits (in our usual notations)

$$Y = \lim_{\Delta t \to 0} \sum_{t_k < T} W_{t_k}(W_{t_{k+1}} - W_{t_k})$$

$$Z = \lim_{\Delta t \to 0} \sum_{t_k < T} W_{t_{k+1}}(W_{t_{k+1}} - W_{t_k})$$

Either show that $Y = Z$ (almost surely) or find a formula for $Z - Y$.

16. Suppose that there are two urns with M_t and N_t balls respectively at time t. Suppose that new balls arrive at the M urn as a poisson process with rate λ. Suppose that each ball in the M urn jumps to the N urn in time dt with probability μdt. Suppose that the N urn has a *server* who removes a ball from the N urn in time dt with probability rdt if there are any balls in the N urn. Write a pair of stochastic differential equations that model this (M_t, N_t) process in the case M_t and N_t are large.

17. Suppose $dX_t = \mu X_t dt + \sigma X_t dW_t$ and that $u(x,t)$ is the probability density of X_t. Write the partial differential equation that u satisfies.

18. A mean reverting geometric Brownian motion is defined by a system of the equations: $dX_t = \mu(X_t, \overline{X}_t) X_t dX_t + \sigma X_t dW_t$, and $d\overline{X}_t = \lambda(X_t - \overline{X}_t)dt$. Suppose we want to calculate $f(x, \overline{x}, t) = E_{x,\overline{x},t}[V(X_T)]$. Write the partial differential equation satisfied by f.