Probability prerequisites for Stochastic Calculus, G63.2902.

Stochastic Calculus assumes a prior, calculus-based course in probability. For example: you must be comfortable working with probability densities, integrating to get means and variances, computing conditional probabilities, etc. You should be able to do this with "multivariate" random variables given by a joint probability density in more than one dimension, computing marginal and conditional probability densities, means and conditional means, covariances, etc. You should undertand the law of large numbers and the central limit theorem and be able to apply them. Independence of random variables and Bayes' rule play a big role.

A widely used text covering this material is the book Elementary Probability Theory by K-L Chung and F. Aitsahlia, 4th edition, Springer, 2003 (chapters 1-7). The 3rd edition is almost equivalent (same title; Chung is the only author; the 4th edition differs mainly by having two final chapters on finance, which aren't needed for Stochastic Calculus). Another good, very concrete book is: Introduction to Probability by D. Bertsekas and J. Tsitsiklis, 2nd edition, Athena Scientific, 2008 (chapters 1-4 and 6). The 2nd edition differs from the 1st mainly by including two chapters on statistical inference (not needed for Stochastic Calculus, but useful for other classes such as Risk & Portfolio Management with Econometrics). You need not purchase or study these books; if you have another probability text at roughly the same level use that.

Reviewing material is best done by working problems. The books listed above have plenty of problems, as does any good textbook. Do a representative selection of them!

Below is a short selection of problems. If you can't do these, you definitely aren't ready to take Stochastic Calculus.

1. We have a container with 300 red balls and 600 blue balls. We mix the balls well and choose one at random, with each ball being equally likely to be chosen. After each choice, we return the chosen ball to the container and mix again.

a. What is the probability that the first n balls chosen are all blue?

b. Let N be the number of blue balls chosen before the first red one. What is the $P(N = n)$? What are the mean and variance of N. Explain your answers using the formulae

$$\begin{aligned} \sum_{n=0}^{\infty} x^n &= \frac{1}{1-x} \qquad \text{for } |x| < 1 \\ \sum_{n=0}^{\infty} n x^n &= x \frac{d}{dx} \frac{1}{1-x} \quad \text{for } |x| < 1 \\ &\text{etc.} \end{aligned}$$

c. What is the probability that $N = 0$ given that $N \leq 2$?

d. What is the probability that N is an even number? Count 0 as an even number.

2. A tourist decides between two plays, called "Good" (G) and "Bad" (B). The probability of the tourist choosing Good is $P(G) = 10\%$. A tourist choosing Good likes it (L) with 70% probability ($P(L \mid G) = .7$) while a tourist choosing Bad dislikes it with 80% probability ($P(D \mid B) = .8$).

a. Draw a probability decision tree diagram to illustrate the choices.

b. Calculate $P(L)$, the probability that the tourist liked the play he or she saw.

c. If the tourist liked the play he or she chose, what is the probability that he or she chose Good?

3. A "triangular" random variable, X, has probability density function (PDF) $f(x)$ given by

$$f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

a. Calculate the mean and variance of X.

b. Suppose X_1 and X_2 are independent samples (copies) of X and $Y = X_1 + X_2$. That is to say that X_1 and X_2 are independent random variables and each has the same density f. Find the PDF for Y.

c. Calculate the mean and variance of Y without using the formula for its PDF.

d. Find $P(Y > 1)$.

e. Suppose X_1, X_2, ..., X_{100} are independent samples of X. Estimate $Pr(X_1 + \cdots + X_{100} > 34)$ using the central limit theorem. You will need access to standard normal probabilities either through a table or a calculator or computer program.

4. Suppose X and Y have a joint PDF

$$f(x,y) = \frac{1}{8\pi} \begin{cases} 4 - x^2 - y^2 & \text{if } x^2 + y^2 \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

a. Calculate $P(X^2 + Y^2 \le 1)$.

b. Calculate the marginal PDF for X alone.

c. What is the covariance between X and Y?

d. Find an event depending on X alone whose probability depends on Y. Use this to show that X is not independent of Y.

e. Write the joint PDF for $U = X^2$ and $V = Y^2$.

f. Calculate the covariance between X^2 and Y^2. It may be easier to do this without using part e. Use this to show, again, that X and Y are not independent.