Assignment 1, due September 19

Corrections: (none yet)

1. In the example of four independent coin tosses, let B_1 be the event of two or fewer H tosses and B_2 be the event of more than 2 H tosses. Assume that each of the 16 outcomes has the same probability.
   (a) Calculate $P(B_1)$ and $P(B_2)$.
   (b) Let X be the total number of H tosses out of the four. Calculate $E[X \mid B_1]$ and $P[X \mid B_2]$.
   (c) Given that $E[X] = 2$, check that the law of total probability is correct in this case.

2. Let (X, Y, Z) be a three dimensional Gaussian random variable with mean zero and covariance matrix

$$\Sigma = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

   This problem gives two approaches to conditioning of multivariate Gaussian random variables. The first is ANOVA (which stands for "analysis of variance"). You figure out how much of the variation of X and Y can be "explained" by Z and how much of the uncertainty is left. The second just used the joint density function of $u(x, y, z)$ and restricts to a specific value of z. If you do it right, the answers will be the same. The algebra is easy because $\det(\Sigma) = 1$ and Σ^{-1} is a matrix of simple integers. Let B be the event $Z = 1$ and let C be the event $Y + Z = 1$.

   (a) Find coefficients α and β so that

$$\left\{ \begin{array}{lcl} X & = & \alpha Z + \widetilde{X} \, , \\ Y & = & \beta Z + \widetilde{Y} \, , \end{array} \right. \tag{1}$$

   with $E[\widetilde{X} Z] = \text{cov}(\widetilde{X}, Z) = 0$, and similarly for $\widetilde{Y}$.

   (b) Calculate $\text{var}(\widetilde{X})$, $\text{var}(\widetilde{Y})$, and $\text{cov}(\widetilde{X}, \widetilde{Y})$ using (1), your formulas for α and β, and the variances and covariances of (X, Y, Z), the entries of Σ. For example,

$$\begin{aligned} E\left[\widetilde{X}^2\right] &= E\left[(X-\alpha Z)^2\right] \\ &= E\left[X^2\right] - 2\alpha E\left[XZ\right] + \cdots \\ &= \Sigma_{xx} - 2\alpha\Sigma_{xy} + \cdots \end{aligned}$$

(c) Using the representation (1) and the results of part (b), it is easy to figure out the distribution of X and Y conditional on a given value of Z. Use this to describe the distribution of (X, Y) conditional on B.

(d) Plug in to check that your answers to part (b) satisfy $\text{var}(X) = \alpha^2\text{var}(Z) + \text{var}(\widetilde{X})$. This represents the variance of X as the sum of the part "explained" by Z and the remaining unexplained variance.

(e) Use a process like parts (a) - (c) to figure out the conditional distribution of X given events B and C. As before, we know that the conditional X distribution is Gaussian, what we don't know before doing the algebra are the parameters – mean and variance.

(f) Write the probability density for (X, Y, Z). Use this to get the conditional probability density $u_B(x, y) = u(x, y \mid B)$ of (X, Y) conditional on the event B? The explicit formula for u gives an explicit formula for u_B, one that involves a certain 2×2 matrix in the exponent.

(g) Find the inverse of this matrix to get the conditional means, variances, and covariances of (X, Y) conditioned on B. Your answer should agree with that in part(c).

(h) Modify steps (f) and (g) to get the conditional density $u_B(x \mid C)$? This is a one dimensional Gaussian whose mean and variance should agree with part (e).

(i) Do the conditioning in one step, that is, find the distribution of X conditioned on $B \cap C$. Check that this agrees with your answer to part (h).